Musculoskeletal Biomechanics

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CMSR Course
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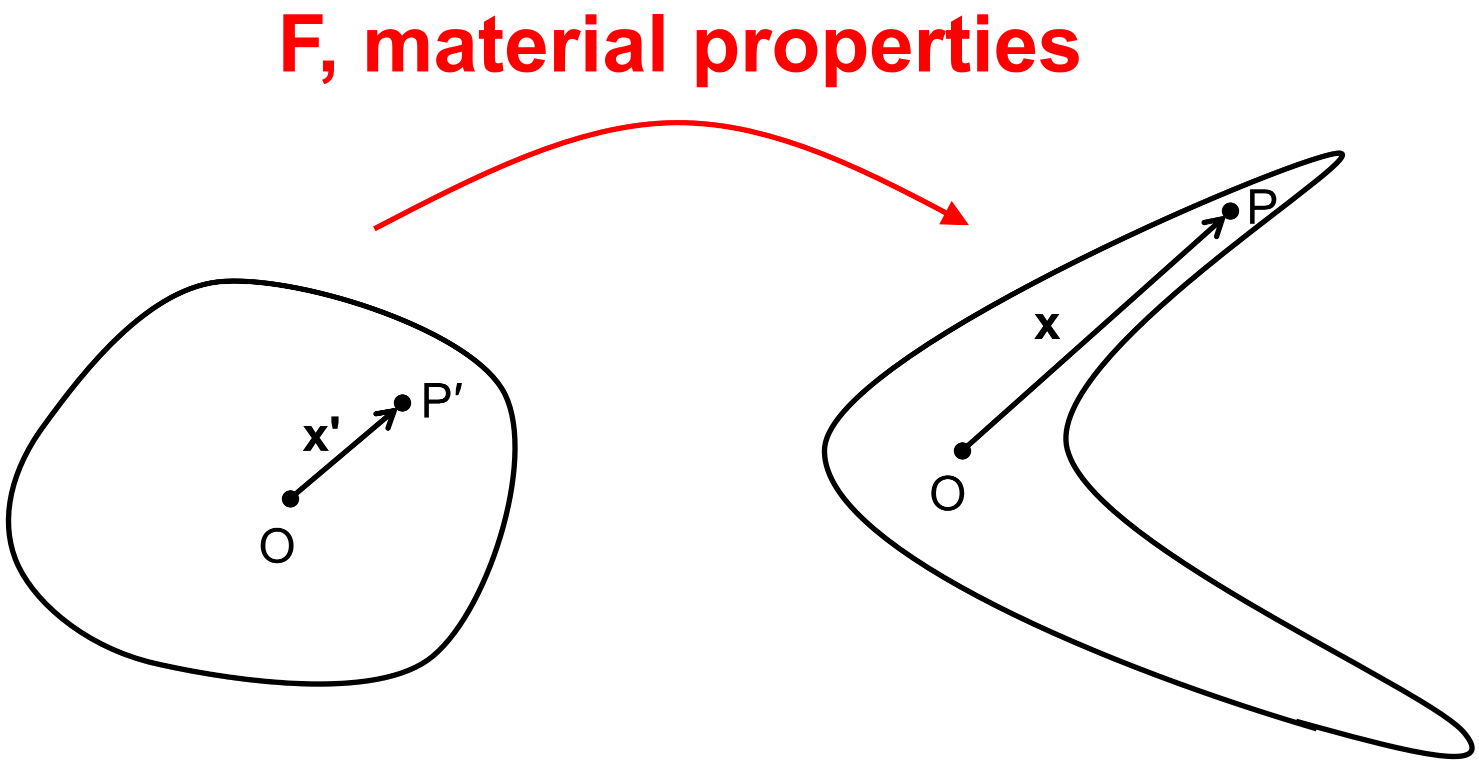
Mechanics

- **Definition of “mechanics”**
  - The study of how solid materials deform, move or break under the action of applied forces



Applications of mechanics in musculoskeletal research

- Determining the optimal material properties of a meniscal implant to minimize risk of cartilage damage
- Monitoring the material properties of a repaired ACL over time to guide clinical care (e.g., when weight bearing may be resumed)
- Computing joint stresses in fetuses at risk of hip dysplasia due to oligohydramnios to link mechanical loading and joint morphogenesis
- Assessing how the transcriptional and metabolic activity of osteocytes is altered by mechanical stresses in the bone
- Comparing the material properties of native cartilage to tissue-engineered cartilage to assess readiness for *in vivo* use
- Calculating stresses in the hip joint in an individual with femoroacetabular impingement
- Determining whether a drug accelerates Achilles tendon healing after a rupture

Tools for biomechanics research

- Theoretical approaches
  - **Limb/joint scale**: Biostatics and biodynamics
  - **Tissue/cell scale**: Models of tissue and cell mechanics
    - 1D models
    - Continuum models (3D)
    - Tensegrity
- Computational approaches
  - Finite element analysis (FEA)
- Experimental approaches
  - Gait analysis
  - Dynamometry
  - Materials testing
  - Elastography
    - Ultrasound
    - MRI

Example application of mechanics in musculoskeletal research

- **Determining the optimal material properties of a meniscal implant to minimize risk of cartilage damage**
  - Need: What mechanical loads does the knee experience during walking? Tools: Biostatics and biodynamics (theoretical), gait analysis (experimental)
  - Need: What mechanical model is appropriate for tissues in the knee? Tool: Continuum mechanics
  - Need: What are the material properties of these tissues? Tool: Materials testing
  - Need: Given these loading conditions and material properties how does knee cartilage deform? Tools: Finite element analysis (FEA)

Example scenario

- **Determining whether a drug accelerates Achilles tendon healing after a rupture (human study)**
- ***Assumption: Material properties of the Achilles tendon are a signature (readout) of Achilles tendon healing***
  - Need: What mechanical loads does the Achilles tendon experience during a diagnostic exercise? Tools: Biostatics and biodynamics, dynamometry
  - Need: What mechanical deformations does the Achilles tendon experience during a diagnostic exercise? Tool: Ultrasound elastography
  - Need: Given these deformations and loading conditions, what are the material properties of the tendon? Tools: Continuum mechanics, inverse finite element analysis

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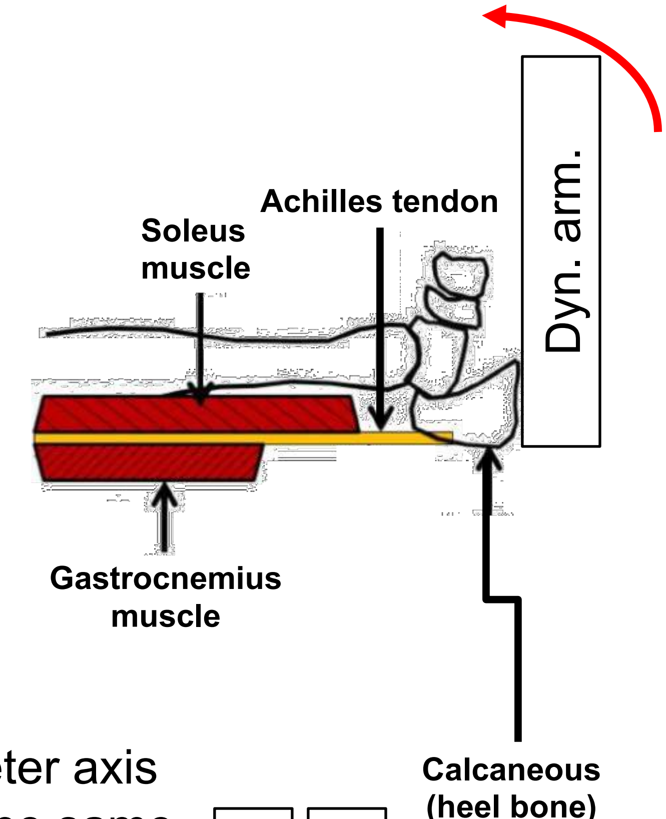
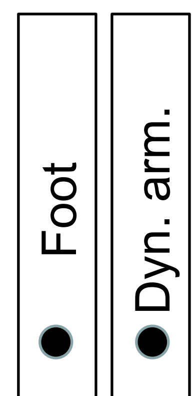
Example approach to indirectly compute *in vivo* forces in a tissue



- The foot is placed on the **dynamometer attachment/armature** such that the dynamometer axis of rotation and the ankle axis of rotation are in the same plane
- Passive dorsiflexion is applied at a constant angular speed such that the angular acceleration $\alpha = 0$



Example approach to <u>indirectly</u> compute *in vivo* forces in a tissue

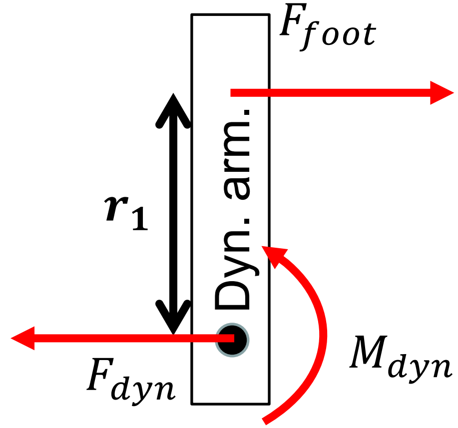
Balance moments (CCW is positive):

$\Sigma M = 0 = r_1 F_{foot} - M_{dyn}$

$\rightarrow \boldsymbol{F_{foot} = M_{dyn}/r_1}$

<u>Note</u>: $\boldsymbol{M_{dyn}}$ is the known moment applied by the dynamometer on the armature (generated by a rotating motor and measured with a torque sensor)

Dynamometer armature free body diagram

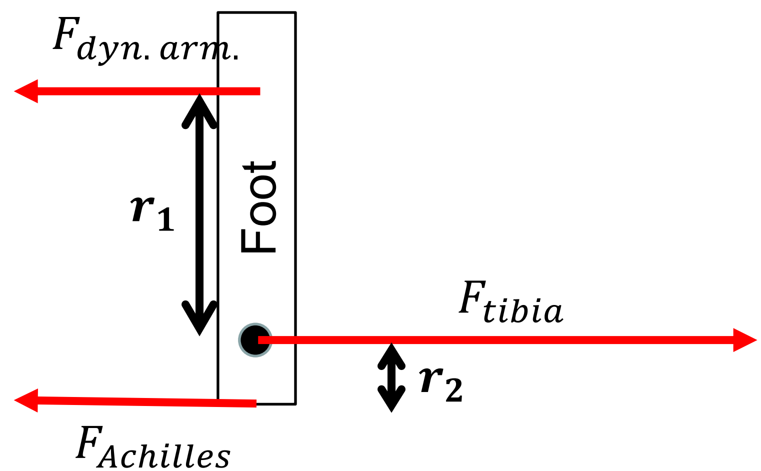


Balance moments:

$\Sigma M = 0 = r_2 F_{Achilles} - r_1 F_{foot}$

$\rightarrow \boldsymbol{F_{Achilles} = F_{foot} r_1 / r_2}$

Foot free body diagram



Example approach to <u>indirectly</u> compute *in vivo* forces in a tissue

- **Comments**
  - Assumptions
    - All other muscle/tendon units and ligaments inactive or disengaged
    - The moment arm of the Achilles tendon (r_2) is known
      - r_2 varies from individual to individual and with ankle angle
      - On average, $r_2 \sim 56\ mm$ at neutral ankle position
  - Added complexities
    - Inertia: If the ankle angle velocity is not constant (as in the case of gait), $\Sigma M = I\alpha \neq 0$.

Example scenario

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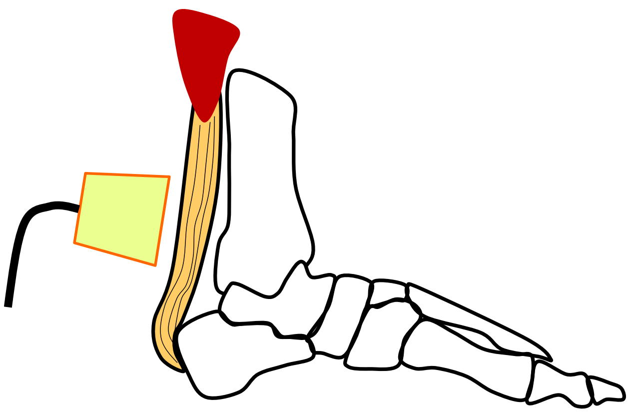
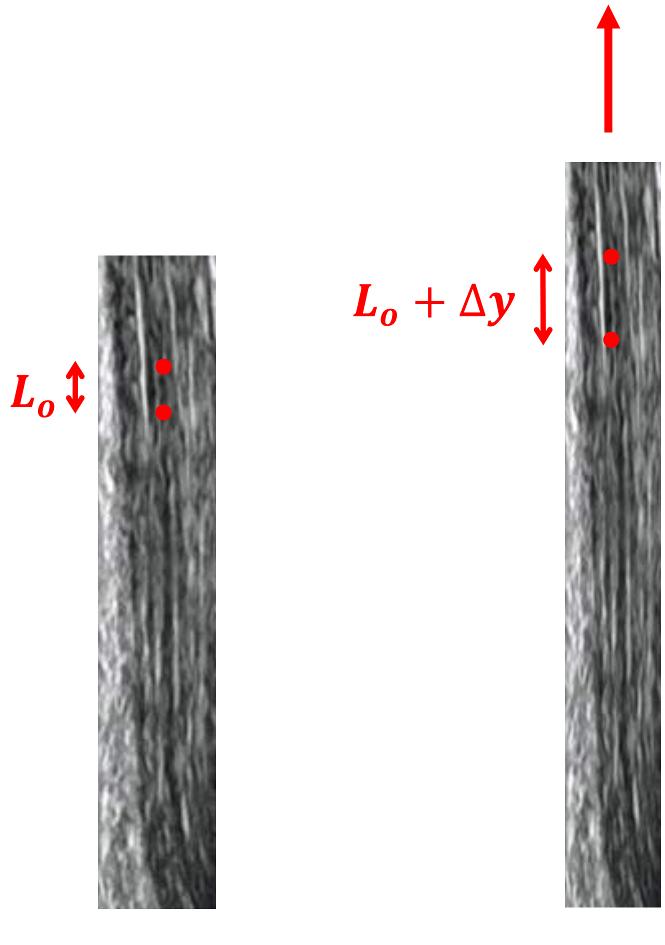
  - Need: What mechanical deformations does the Achilles tendon experience during a diagnostic exercise? Tool: Ultrasound elastography
  - Need: Given these deformations and loading conditions, what are the material properties of the tendon? Tools: Continuum mechanics, inverse finite element analysis

Computing strain in a region of the Achilles tendon





$$\varepsilon = \frac{\Delta y}{L_o}$$

Example scenario

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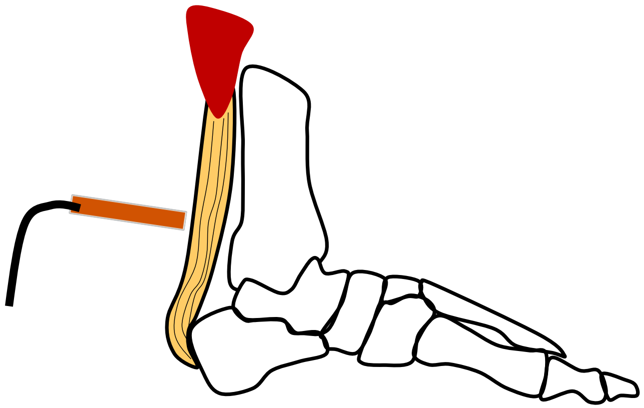
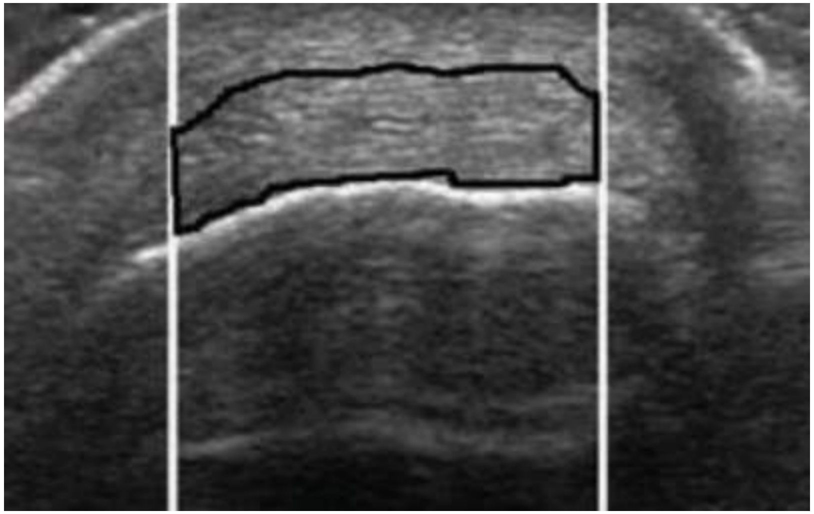
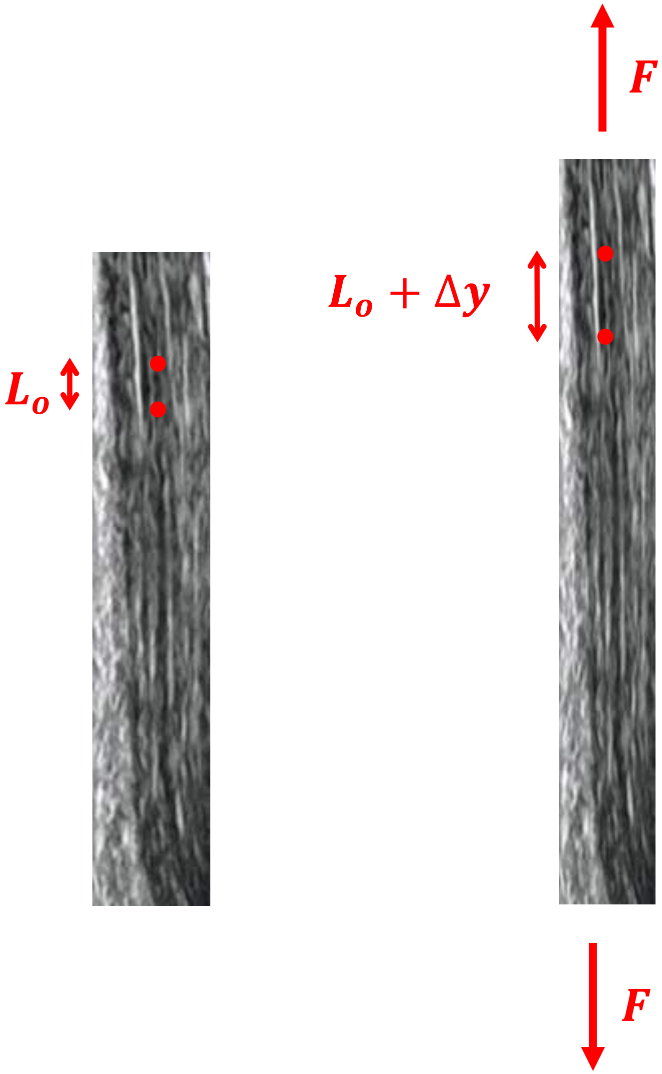
Assessing the cross-sectional area of the Achilles tendon





Enclosed
area = A

If the Achilles tendon were a simple material…



STRAIN $\varepsilon = \frac{\Delta y}{L_o}$

STRESS $\sigma = F/A$

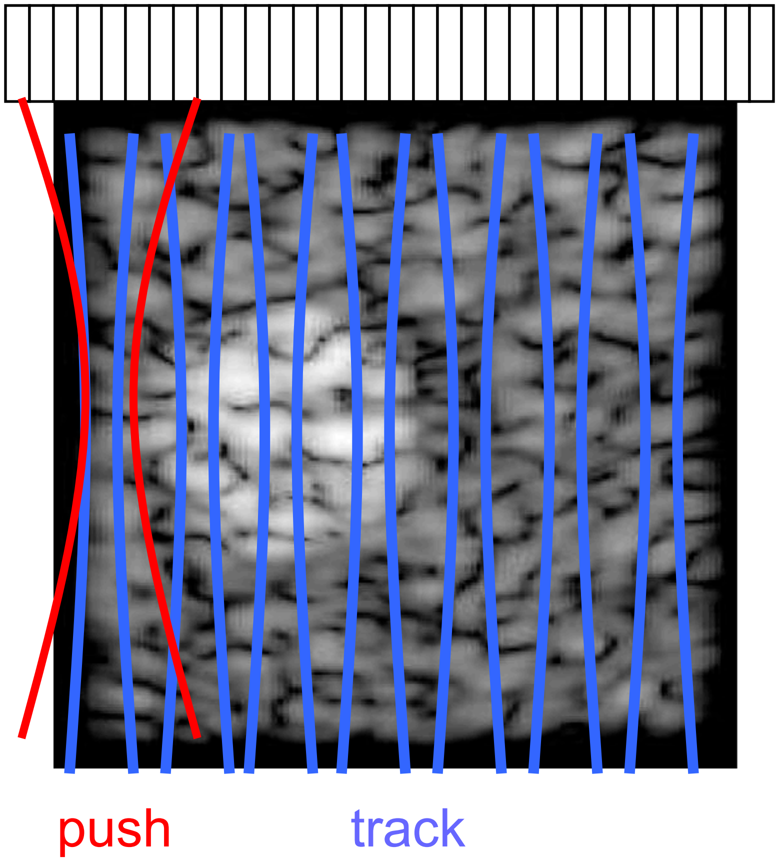
YOUNG'S MODULUS $E = \sigma/\varepsilon$

Material property

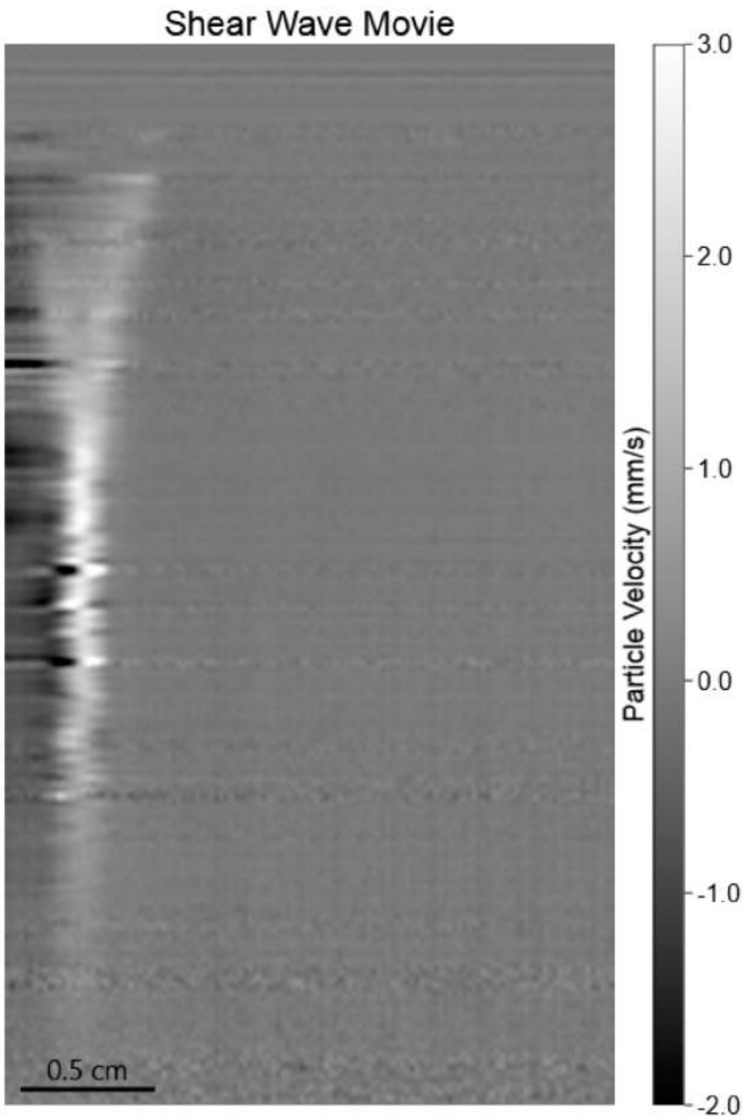
Example scenario

- **Determining whether a drug accelerates Achilles tendon healing after a rupture (<u>human study</u>)**
- ***Assumption: Material properties of the Achilles tendon are a signature (readout) of Achilles tendon healing***
  - What are the material properties of the tendon at a given time point? Tool: Shear wave elastograpy

Shear wave elastography



- Material properties may be estimated from observation of speed and distortion of propagating shear wave (e.g., $v_s = \sqrt{\frac{E}{3\rho}}$ in a simple material under no stress)

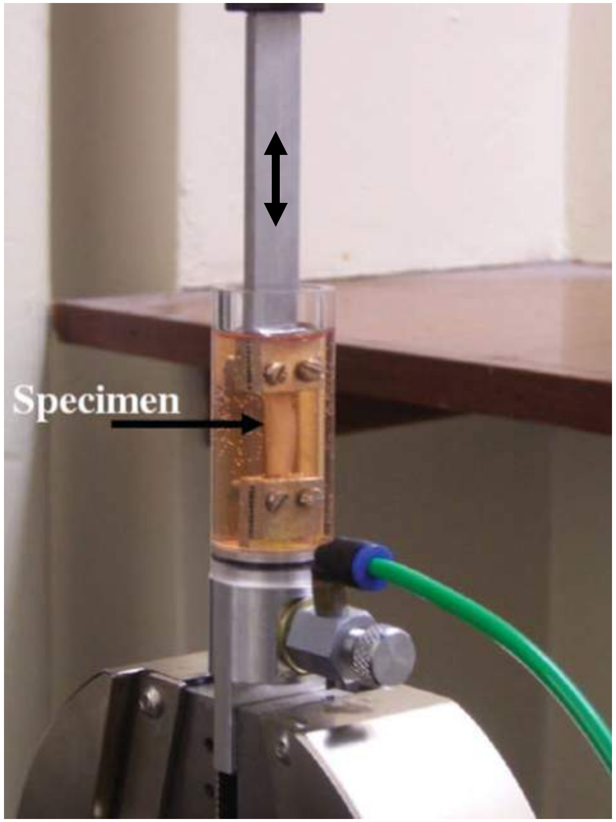


Slide courtesy of S. McAleavey

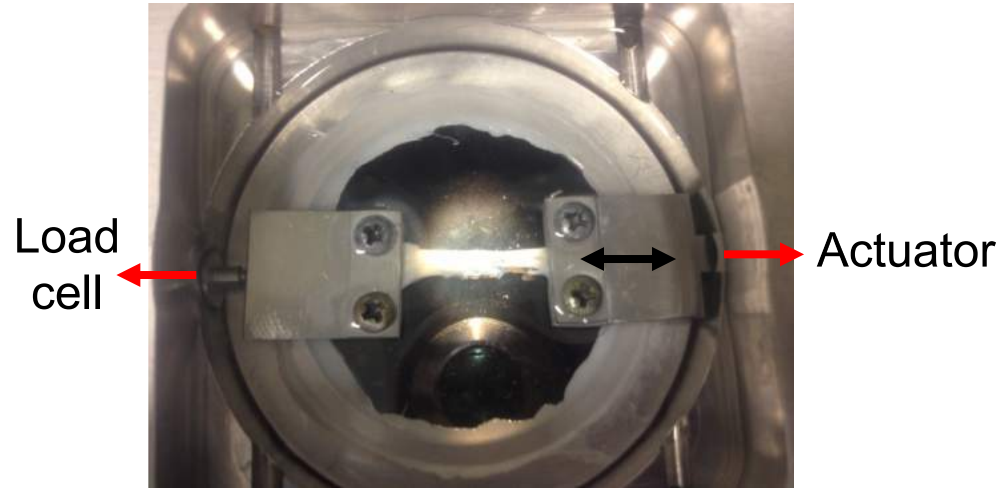
Example scenario (alternative)

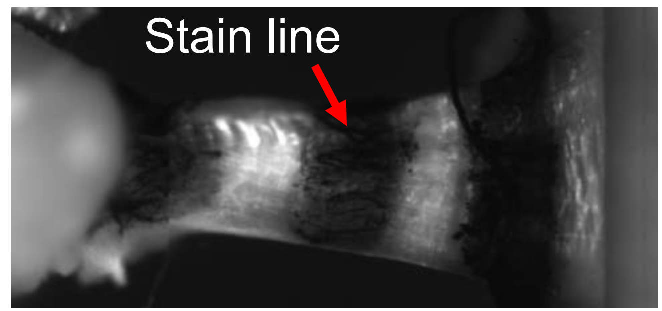
- **Determining whether a drug accelerates Achilles tendon healing after a rupture (<u>animal study</u>)**
- ***Assumption: Material properties of the Achilles tendon are a signature (readout) of Achilles tendon healing***
    - What are the material properties of the tendon at a given time point? Tool: *Ex-vivo* materials testing

Ex-vivo uniaxial tensile test



Elsheikh et al. (2007)

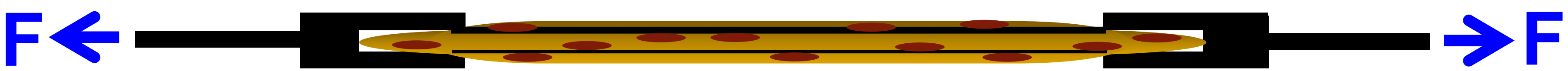




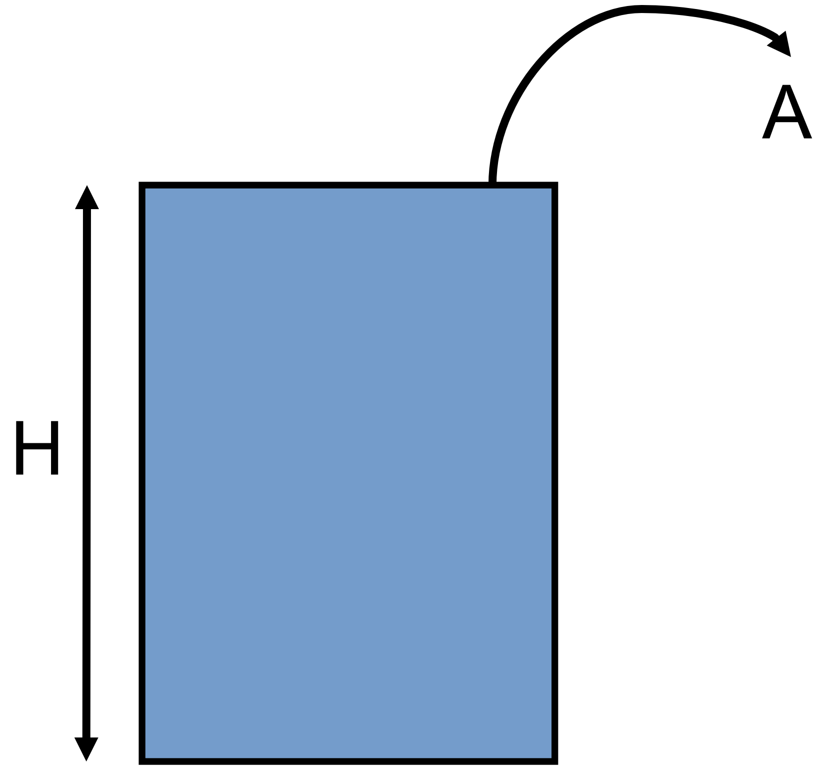
- One grip is coupled to a linear actuator (e.g., a piezoelectic crystal)
- Other grip is coupled to a load cell or other force transducer (could be as simple as a spring of known k whose displacement is measured)
- Tests may be performed with microscope-mounted systems, enabling measurement of local strains and local mechanical properties and avoiding artefacts due to specimen slippage

Gripping strategies

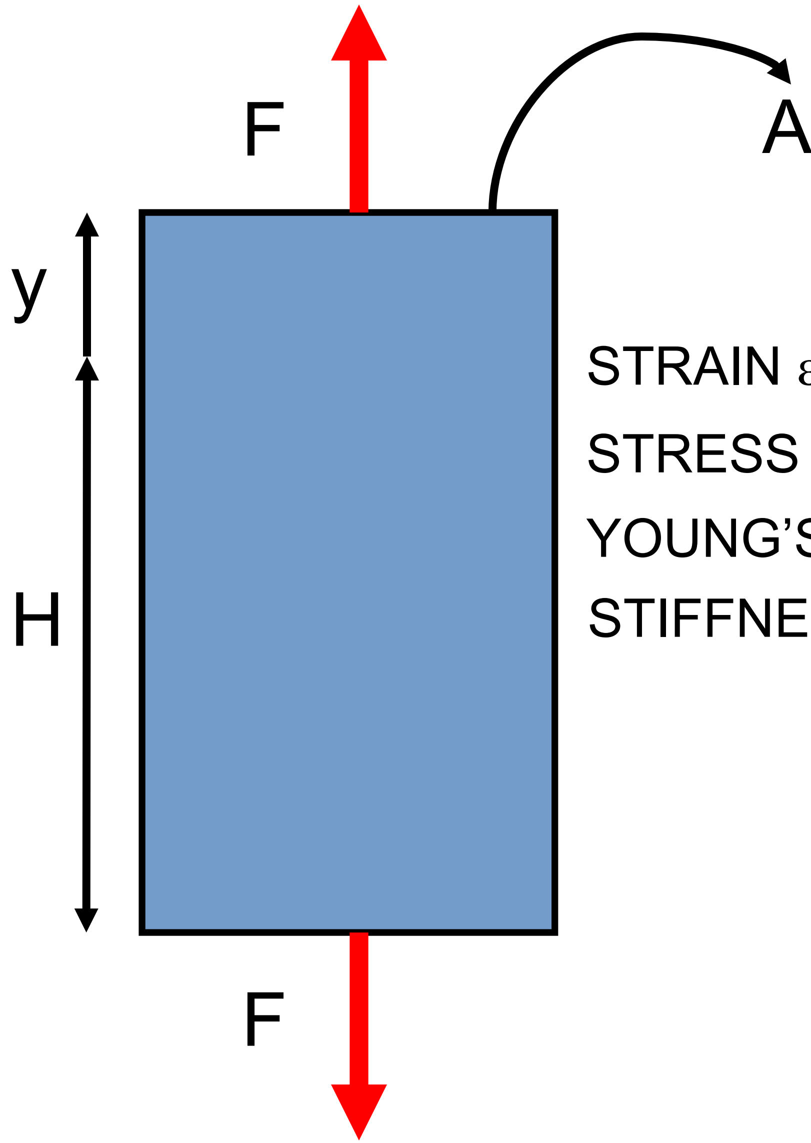
- Screw-tightened clamps or vices (may be serrated for enhanced adhesion)
- Pneumatically-tightened clamps
- Cyanoacrylate (super glue, optimal glue for tissues)
- Sandpaper (used in conjunction with clamps for tension or by itself for shear)
- Freeze clamps (clamps jacketed with liquid nitrogen that adhere to the specimen through freezing)



Tension/extension in 1D



Tension/extension in 1D



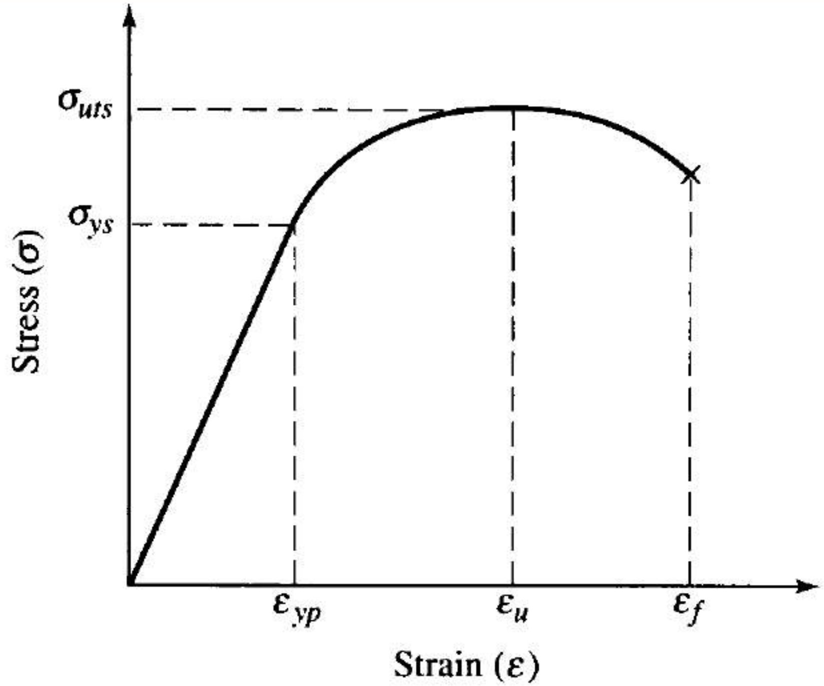
STRAIN $\varepsilon = y/H$

STRESS $\sigma = F/A$

YOUNG'S MODULUS $E = \sigma/\varepsilon$

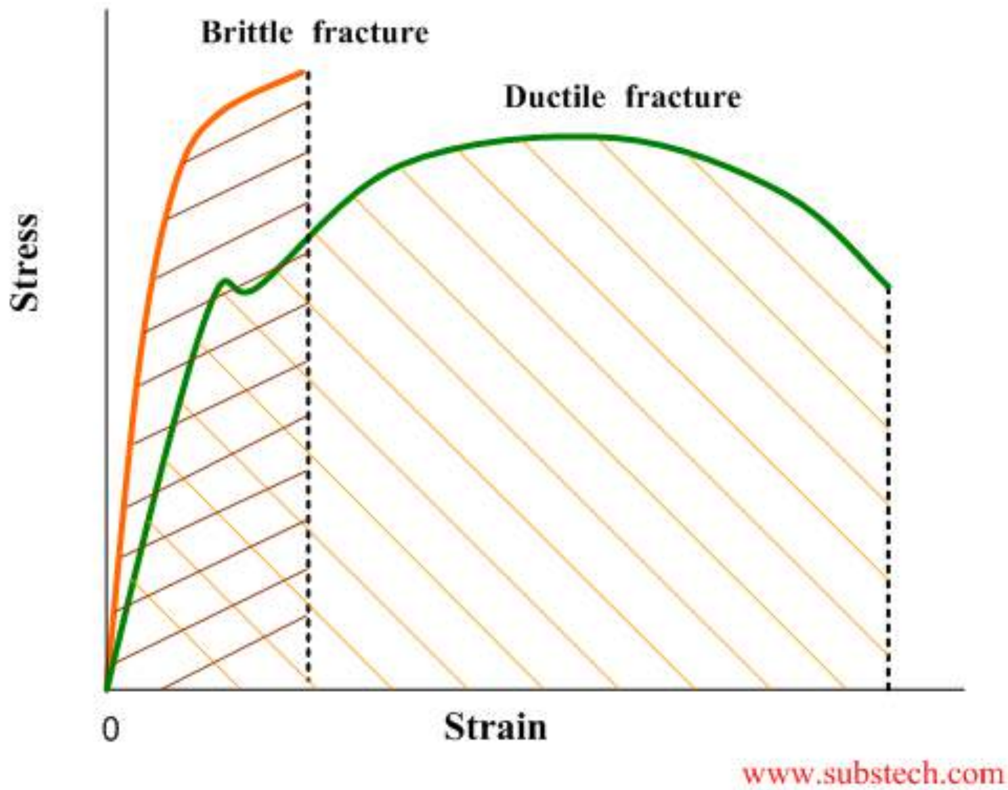
STIFFNESS $k = F/y$

Uniaxial tension/extension: The stress-strain curve



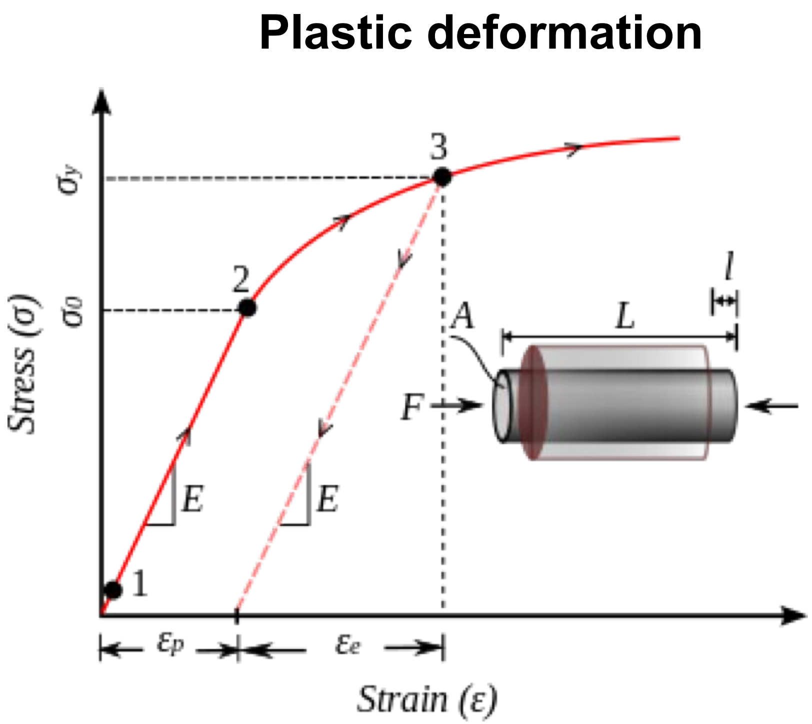
- σ_{ys} **is the <u>yield stress</u>,** ε_{yp} **is the <u>yield strain</u>**
- σ_{uts} **is the <u>ultimate tensile strength</u>,** ε_u **is the <u>ultimate strain</u>**
- ε_f **is the <u>failure strain</u>**
- **Say we obtain the curve above. We now know that σ vs. ε is linear up until the yield point σ_{ys}. E is then most appropriately the slope of σ vs. ε, but is technically also just equal to σ/ε for any σ up to σ_{ys}.**
- **Young's modulus E and σ_{uts} are measured <u>material properties</u>, and $E = \sigma/\varepsilon$ is the empirically-derived <u>constitutive relationship</u>.**

Strength vs. toughness

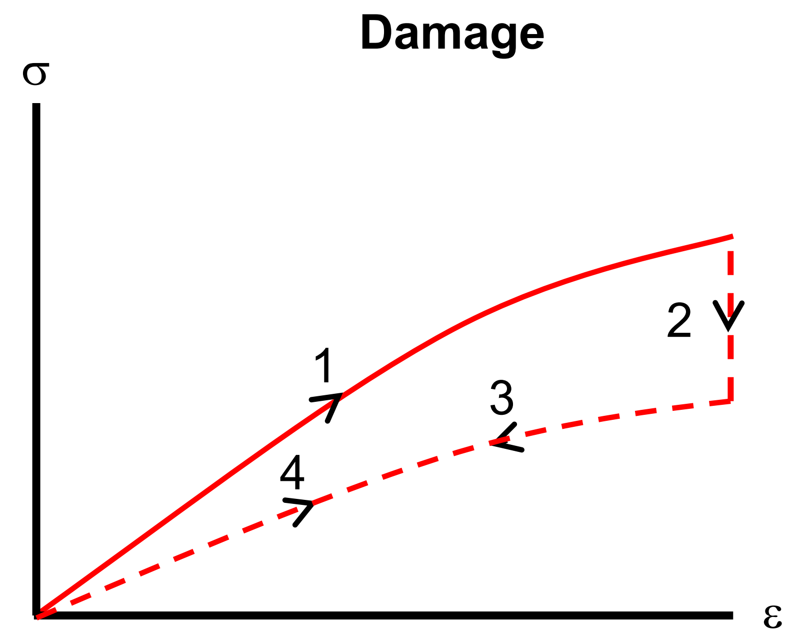


- Toughness = area under stress strain curve (units = energy/volume)
- Toughness is larger for materials with high failure strains
- Tough materials are not always strong (see curve above)

Plastic deformation vs. damage



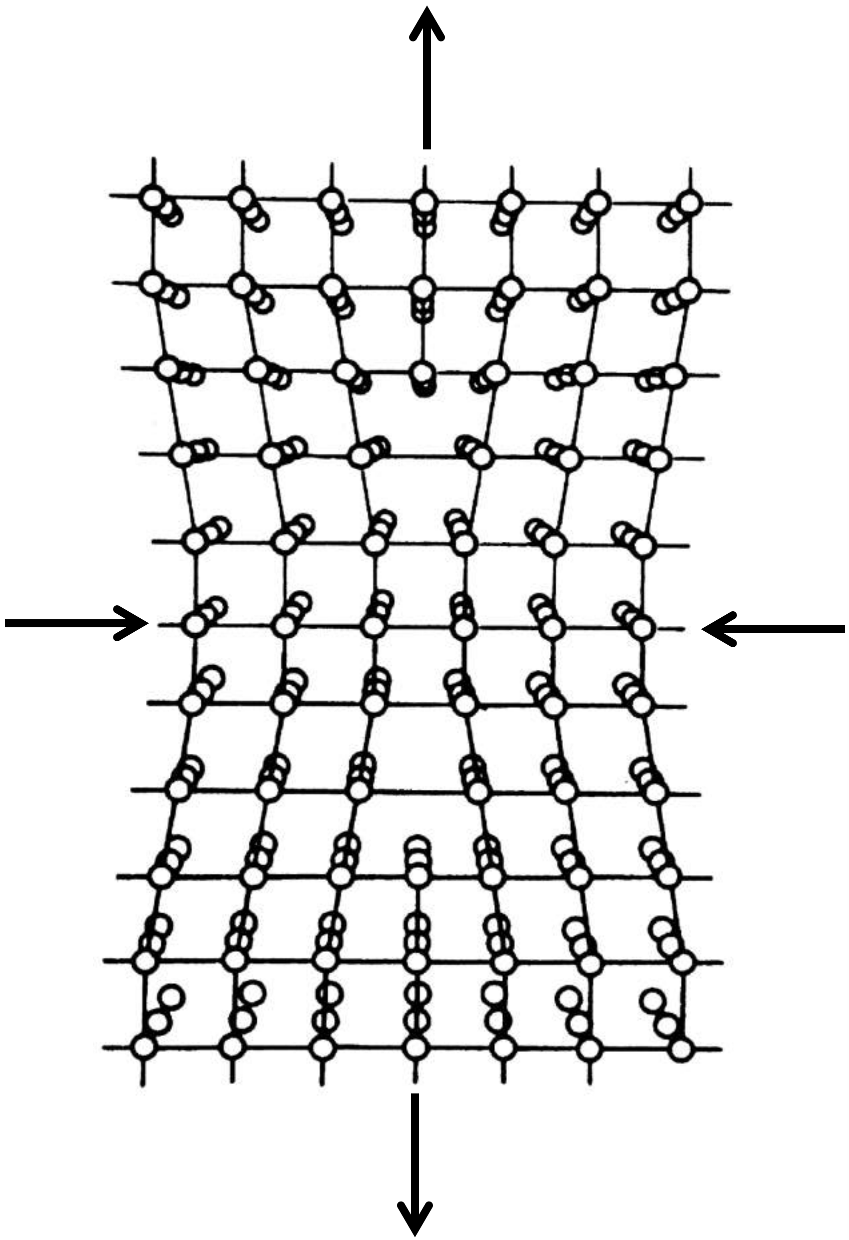
https://en.wikipedia.org/wiki/Work_hardening



- Plastic deformation = permanent deformation in a material
  - Modulus is unaltered
- Damage = reduced modulus

Mechanisms of plastic deformation

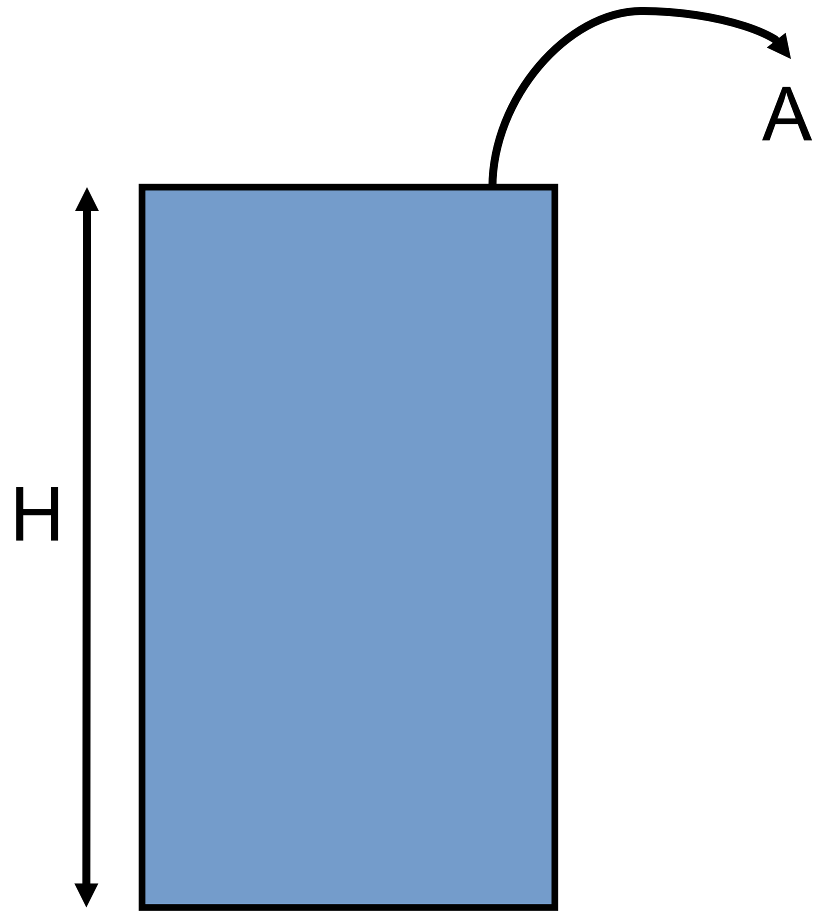
- **Plasticity**
  - **Permanent deformation (due to dislocations—molecular rearrangements that relieve stress, bond breakage, etc.) increases the effective gauge length L_0 of the specimen so that more strain is now required to engage fibers and get a stress response**



Material vs. structural properties

- **Q: What is the difference between <u>material</u> and <u>structural</u> properties (e.g., Young's modulus E vs. stiffness k)?**
  - Structural property ($k=F/y$) changes with specimen size, material property ($E=\sigma/\varepsilon$) doesn't
- **Q: You apply a vertical tensile force of F to two cylindrical aluminum bars of radius R and height H and 2H under uniaxial extension. Which bar has a higher modulus? Which bar is stiffer? Which bar stretches more?**
  - $\varepsilon = y/H$
  - $\sigma = F/A$
  - Modulus of aluminum $E = \sigma/\varepsilon$ = material property of aluminum → $\mathbf{E = E_1 = E_2}$
  - For bar 1, $k_1 = F/y = A\sigma/H\varepsilon = AE/H$
  - For bar 2, $k_2 = F/y = A\sigma/2H\varepsilon = AE/2H$
  - Thus, $0.5k_1 = k_2$ → $\mathbf{k_1 > k_2}$
  - $y_1 = F/k_1$
  - $y_2 = F/k_2 = F/(0.5k_1)$ → $y_2 = 2y_1$ → $\mathbf{y_2 > y_1}$

Compression in 1D



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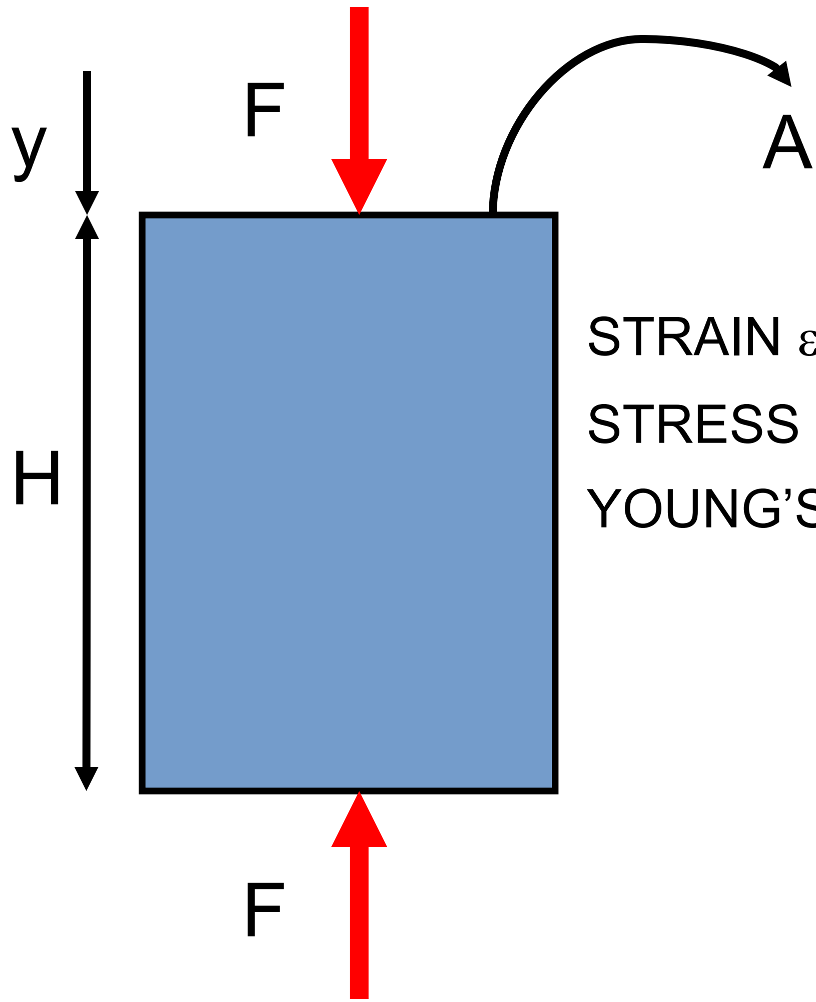
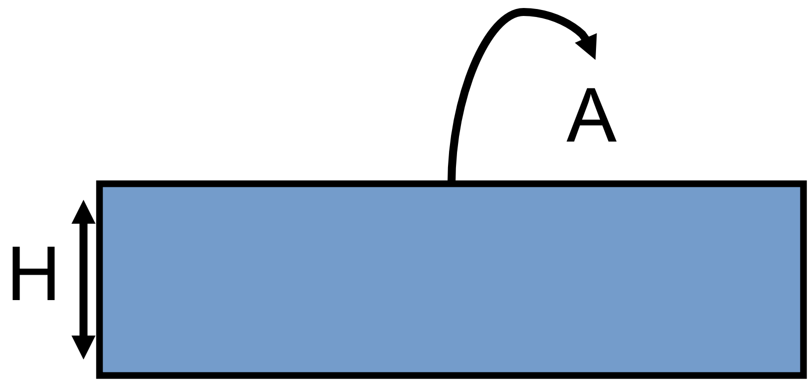


STRAIN $\varepsilon = y/h$

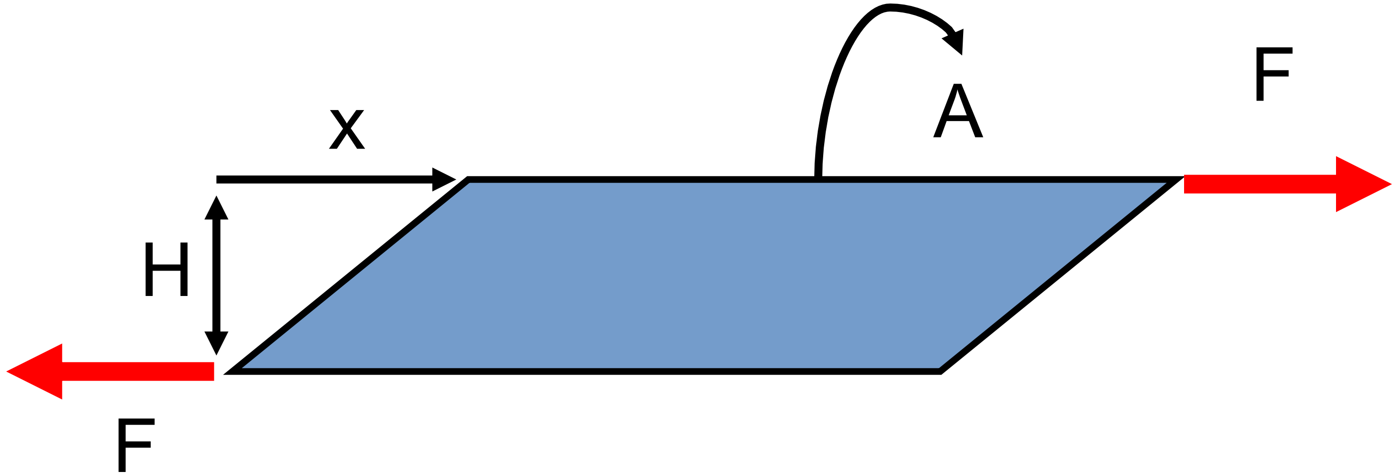
STRESS $\sigma = F/A$

YOUNG'S MODULUS $E = \sigma/\varepsilon$

Simple shear



Simple shear

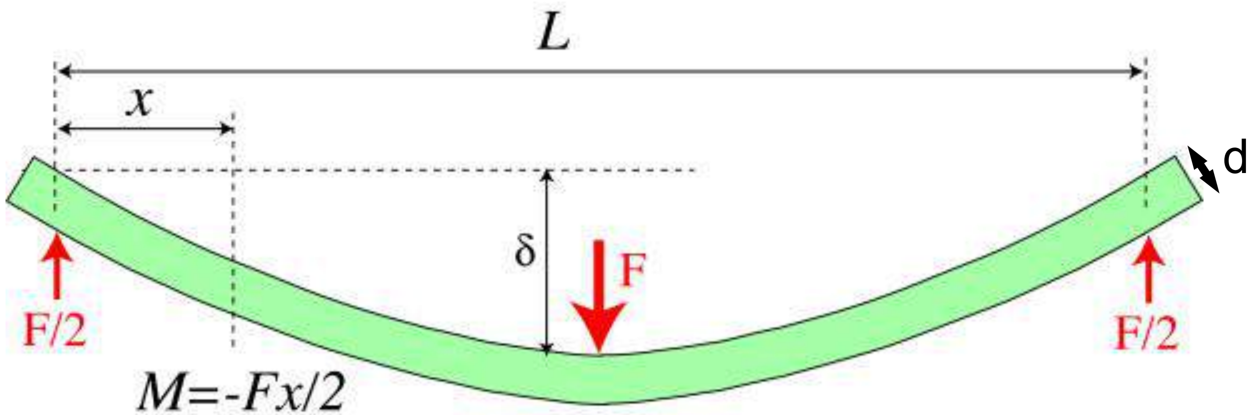


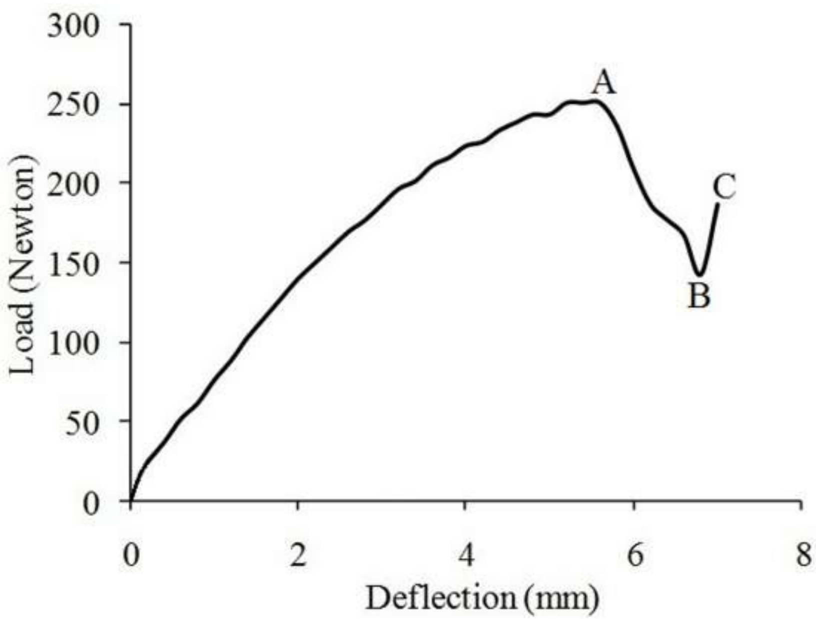
SHEAR STRAIN γ = x/h

SHEAR STRESS τ = F/A

SHEAR MODULUS G = τ/γ

Three-point bending





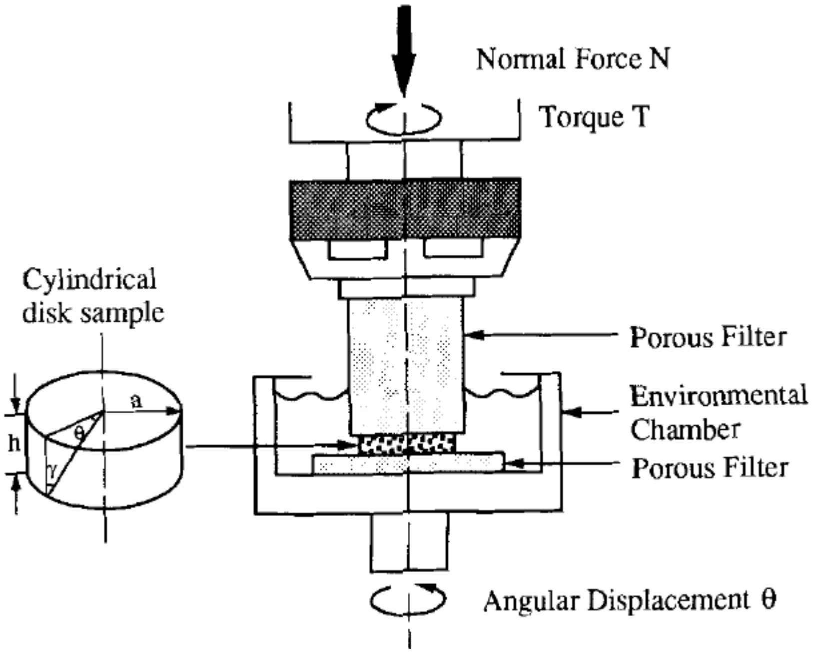
- I = moment of inertia of beam
- F = applied load (concentrated at the center of the beam)
- L = length of beam
- δ_{max} = maximum deflection
- E = modulus of beam
- $\delta_{max} = \frac{PL^3}{48EI}$

Three-point bending

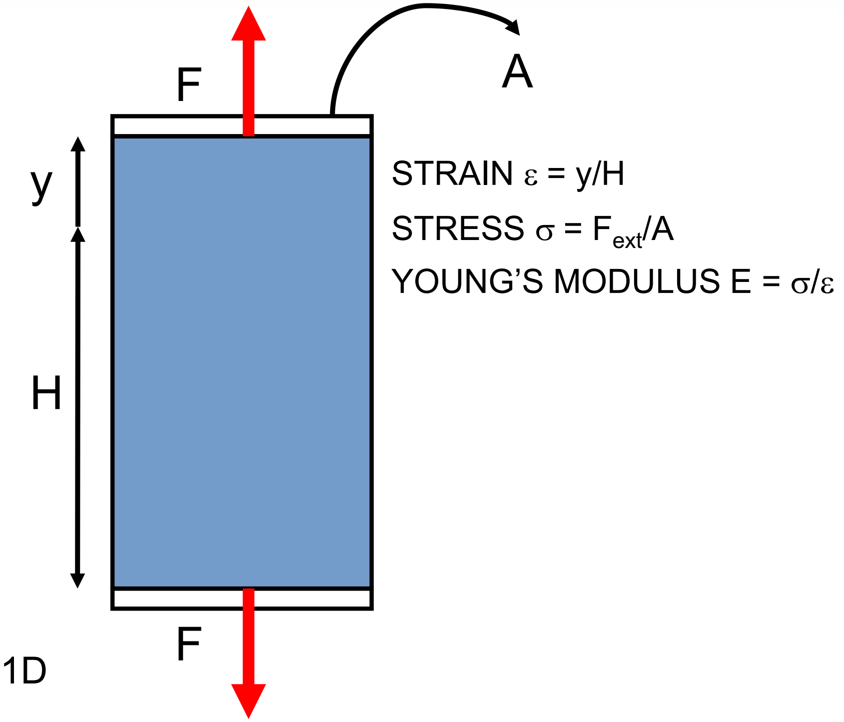
- **3 point bending of rat tibias (University of Minnesota)**

Torsion testing

- From the figure on the right:
  - Shear stress at the edge = $\tau(r=a) = Ta/I_p$
    - I_p is the polar moment of inertia of the specimen = $\pi a^4/2$ for a cylinder = $\frac{\pi}{64}\left({D_0}^4 - {D_i}^4\right)$ for a hollow shell (cortical bone)
    - T is the torque measured by the torque transducer
  - Shear $\gamma(r) = \theta r/h$
  - Shear strain at the edge = $\gamma(r=a) = \theta a/h$
    - The arc length L corresponding to the angle θ on the top surface of the cylinder (see figure) is given by $\theta = L/a$ → $L = \theta a$
    - Similarly, on the side of the cylinder, $\gamma = L/h$ (see the figure) → $L = \gamma h$
    - Thus, $\theta a = \gamma h$ and $\gamma = \theta a/h$
    - Angular displacement θ is controlled by a motor attached to the lower platen
  - Shear modulus = $\tau(r=a) / \gamma(r=a)$ → $G = \frac{Th}{\theta I_p}$



Complications of this simple picture



- **Complex geometry**
  - Biological materials are not always cylinders or rectangular boxes
- **Multidimensionality**
  - Biological materials are 3D, not 1D
- **Nonlinearity**
  - Stress-strain curves of biological materials are almost never straight lines
- **Anisotropy**
  - Young's modulus and other material properties are different along different directions
- **Viscoelasticity/poroelasticity**
  - Mechanical response depends on loading rate, loading history and time
- **Heterogeneity**
  - Mechanical properties vary by location within a cell, tissue or organ
- **Objectivity**
  - For large strains, this definition of strain, known as "infinitesimal strain" is flawed (won't discuss today)

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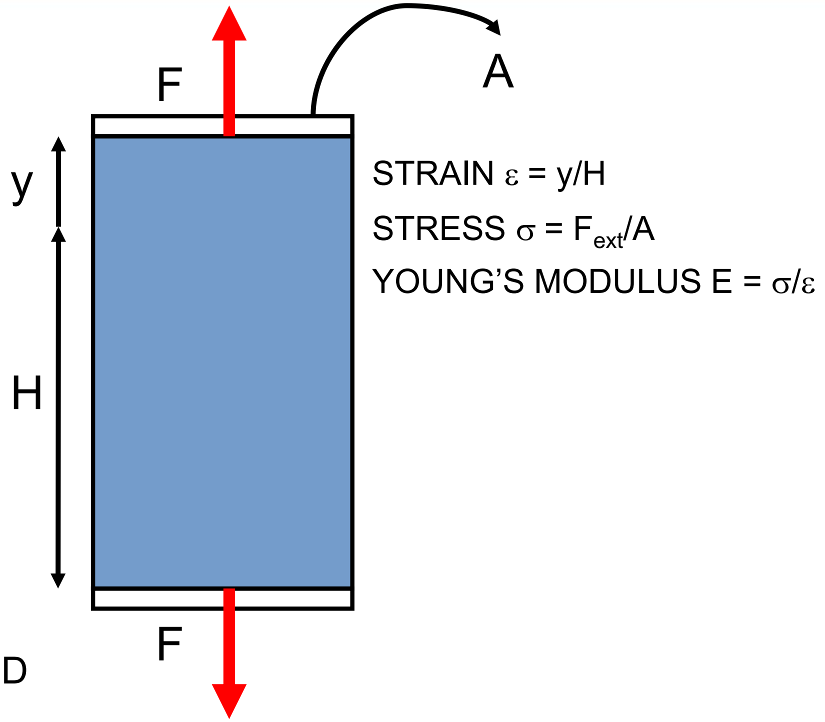

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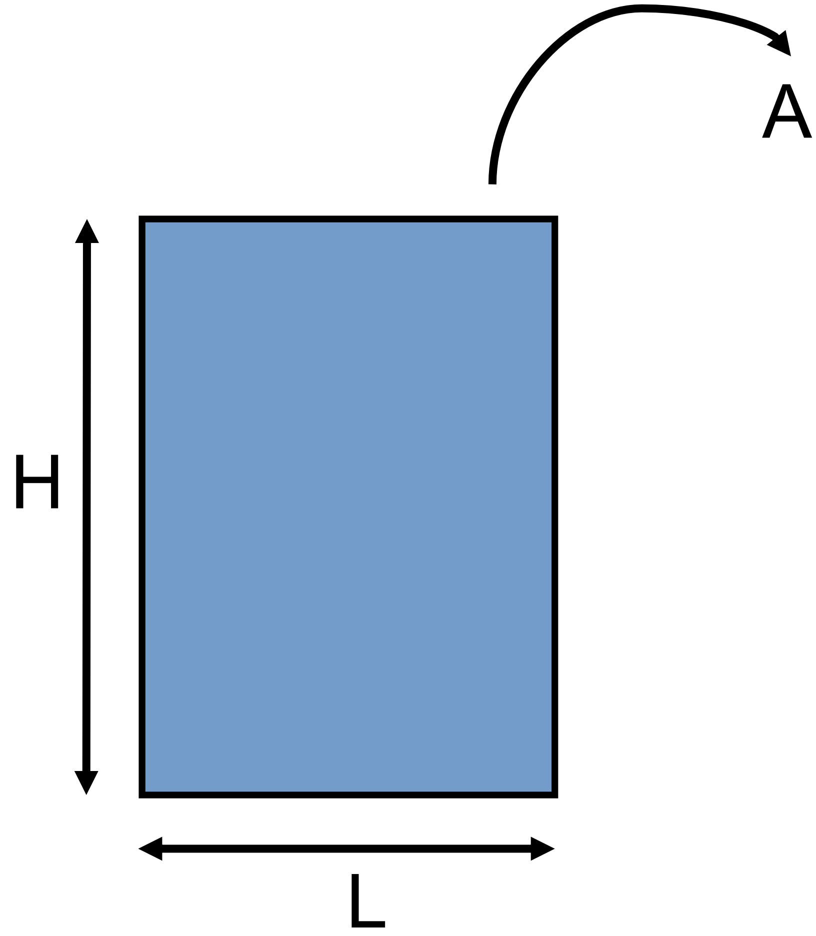
Tension/extension in 2D



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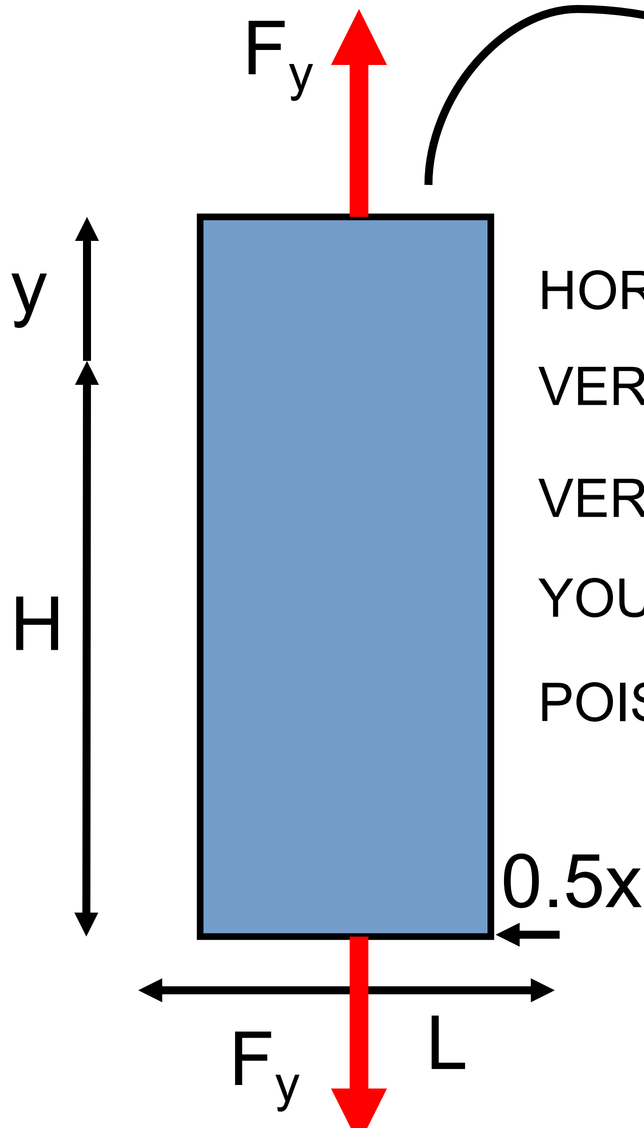


HORIZONTAL STRAIN ε_x = x/L

VERTICAL STRAIN ε_y = y/H

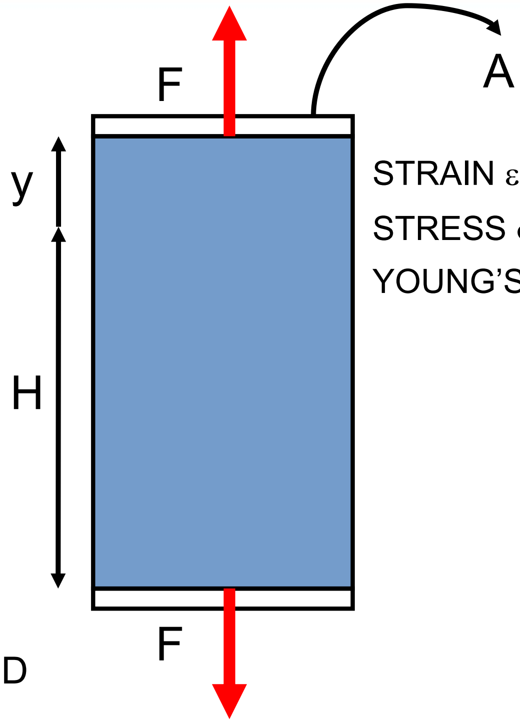
VERTICAL STRESS σ_y = F_y/A

YOUNG'S MODULUS E = σ_y/ε_y

POISSON RATIO = $-\varepsilon_x/\varepsilon_y$

- **In general, we need to consider not only deformation resulting from parallel stresses (e.g., ε_y due to σ_y) but also deformation resulting from transverse stresses (e.g., ε_x due to σ_y).**

Complications of this simple picture



STRAIN $\varepsilon = y/H$

STRESS $\sigma = F_{ext}/A$

YOUNG'S MODULUS $E = \sigma/\varepsilon$

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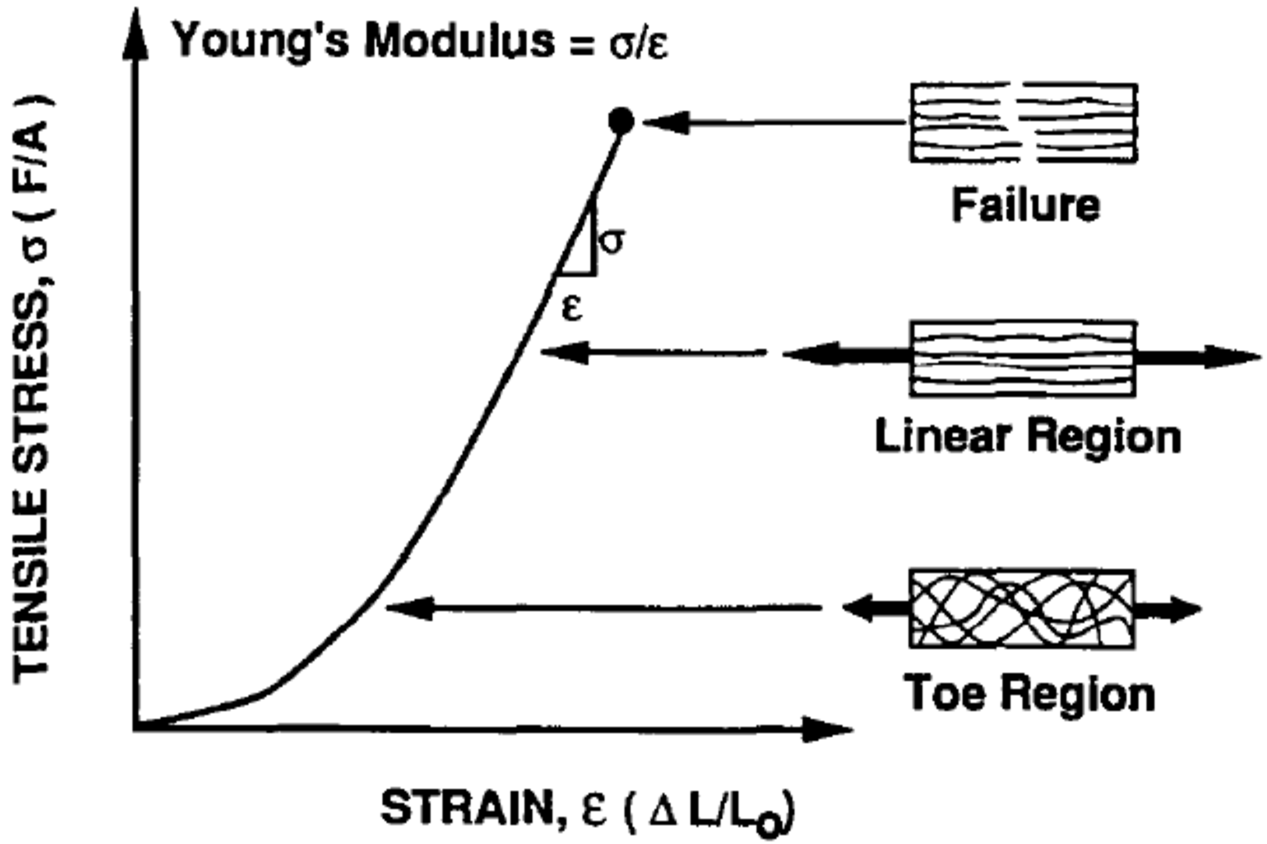
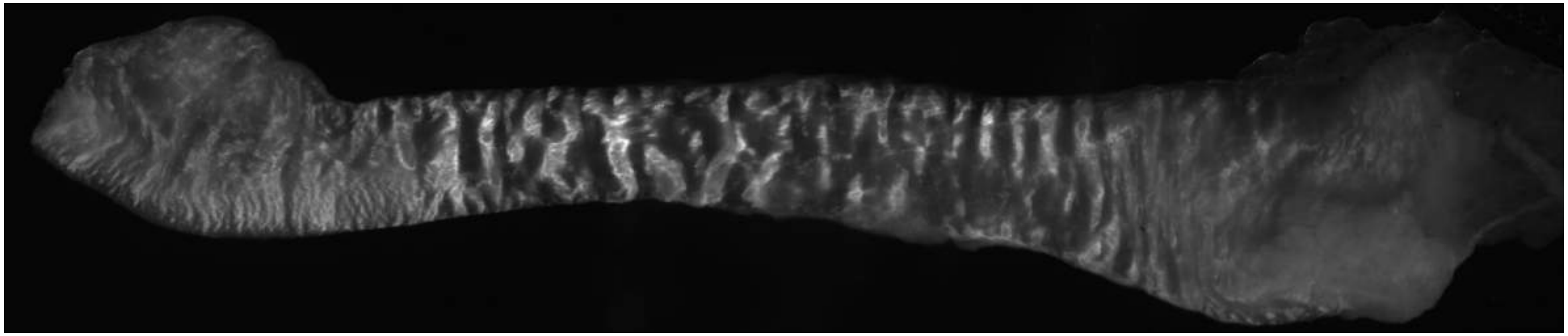
Nonlinearity due to realignment or uncrimping



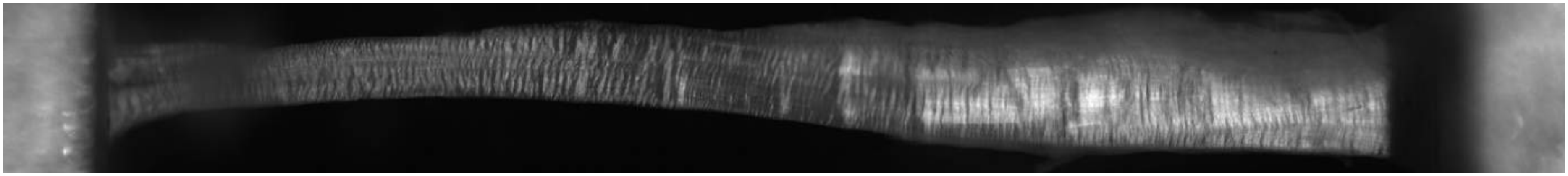
Mow et al. (1991)

Tendon crimp

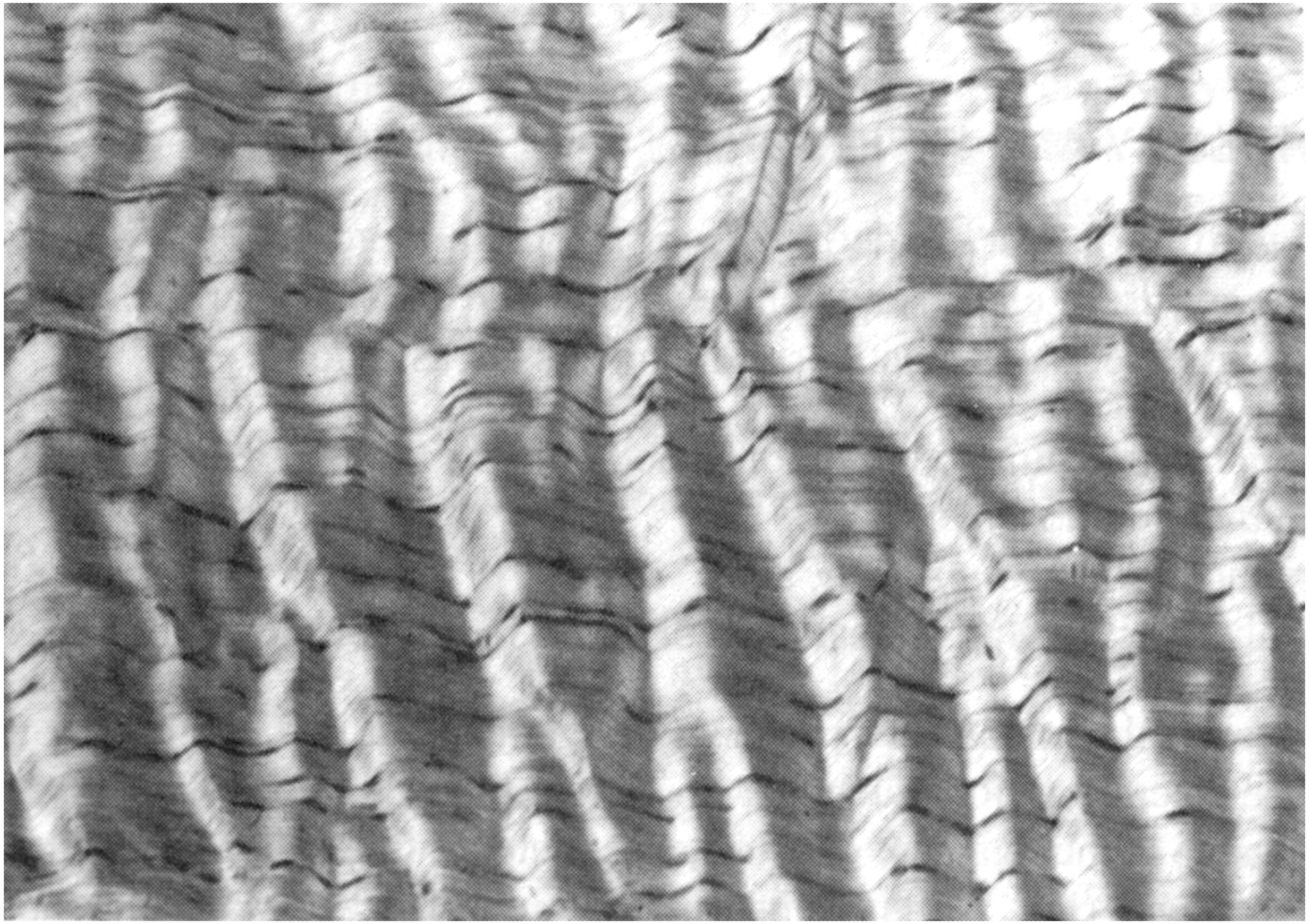
Mouse anterior tibialis tendon



Human Achilles tendon



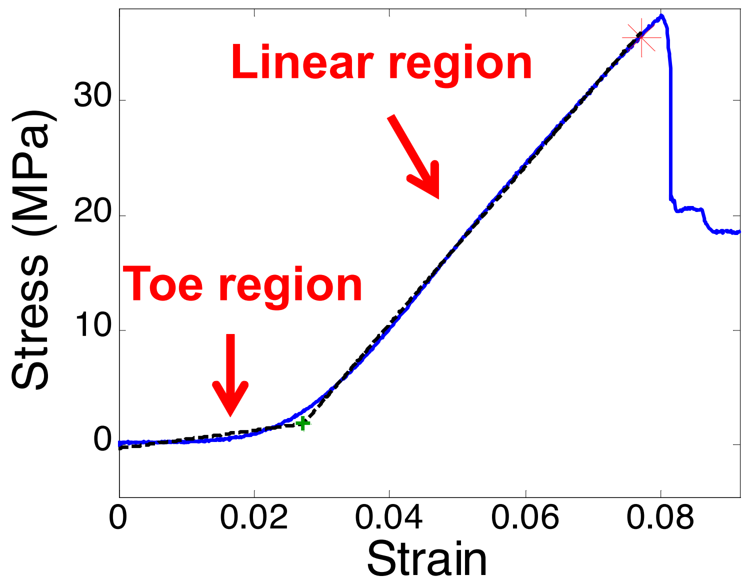
Tendon crimp (SEM)



Nigg and Herzog (1999)

Characterization and modeling of nonlinearity

- One simple approach: Decompose stress-strain curve into two lines (bilinear fit) and compute 2 Young's moduli (toe and linear moduli)

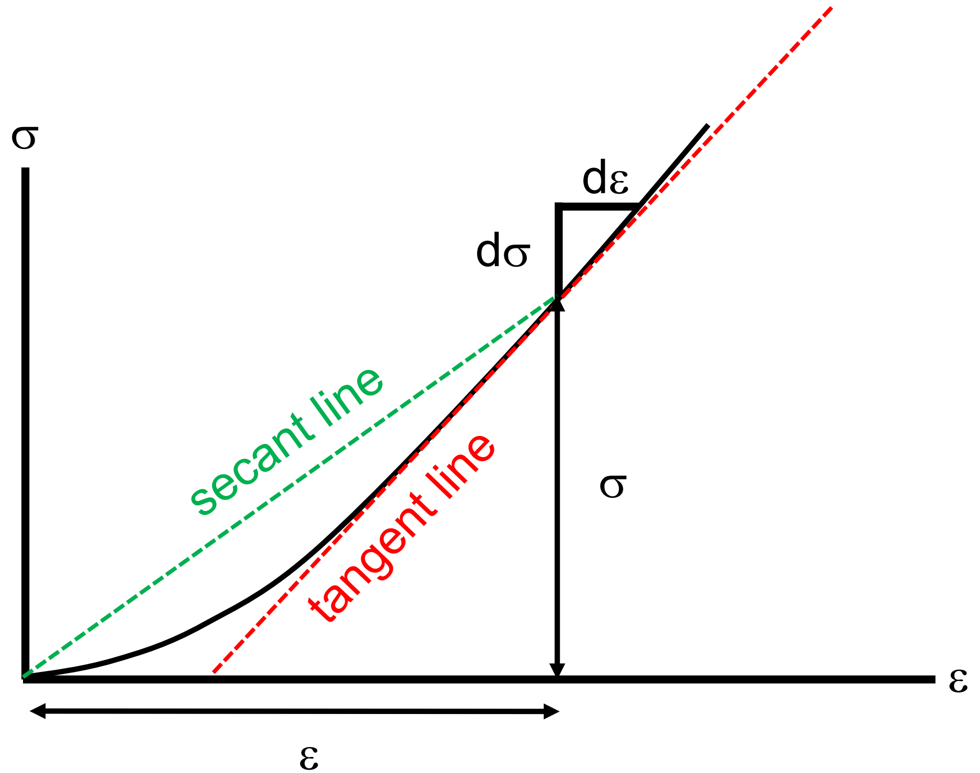


*** Data from murine FCU tendon**
*** Dashed black lines = bilinear fit**

Buckley et al. (2013)

Characterization and modeling of nonlinearity

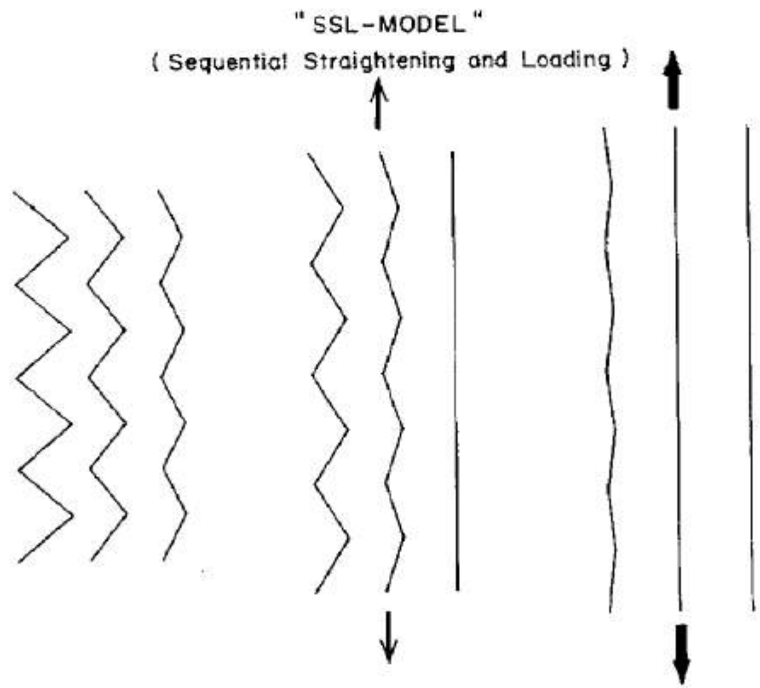
- Another simple approach: Consider E to be a function of ε or σ such that either:
  - $E_{sec}(\varepsilon) = \sigma/\varepsilon$ (secant modulus) or
  - $E_{tan}(\varepsilon) = d\sigma/d\varepsilon$ (tangent modulus)
- What is the disadvantage of this strategy?
  - How can we compare treated and untreated tendons (or other tissues) when our readout parameter (e.g., E_{tan}) is not constant (i.e., depends on ε)? Do we compare E_{tan} at a specific strain? At a few strains?
  - A better approach is to characterize the stress-strain curve with a small number of parameters

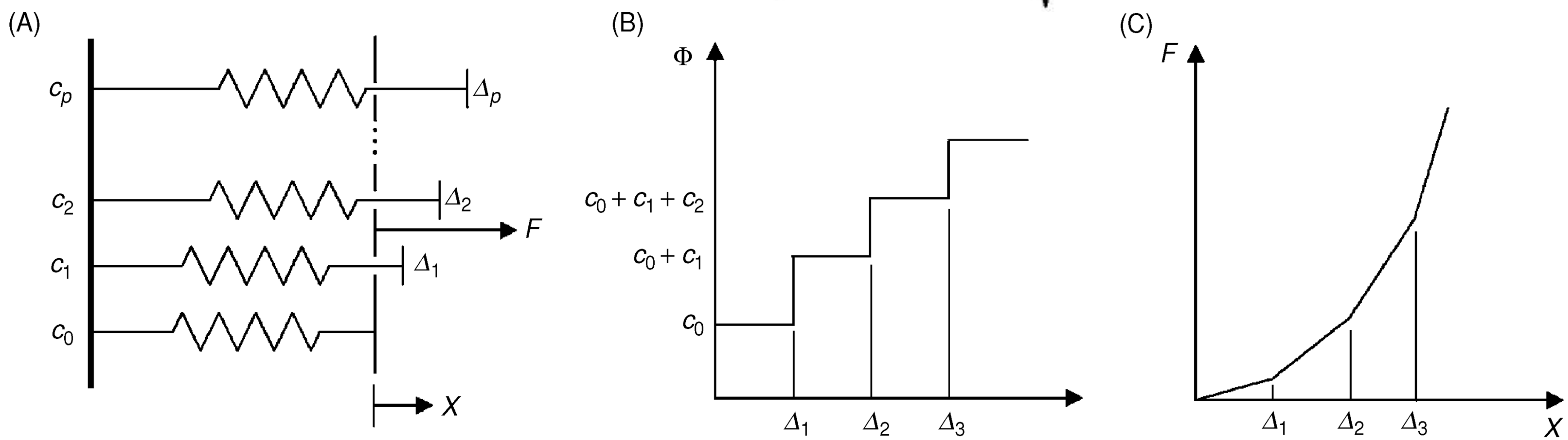


Characterization and modeling of nonlinearity

- *Alternative approach: Fit 2+ parameter function to σ vs. ε (only requires 2+ material properties instead of 1 at each strain)*
- Example 1: $\boldsymbol{\sigma = c_1\varepsilon + c_2\varepsilon^2}$
  - $E_{secant} = \sigma/\varepsilon = c_1 + c_2\varepsilon \rightarrow E_{tangent} \propto \varepsilon$
  - $E_{tangent} = d\sigma/d\varepsilon = c_1 + 2c_2\varepsilon \rightarrow E_{tangent} \propto \varepsilon$
  - So what does c_1 represent?
    - Secant or tangent modulus at $\varepsilon = 0$
- Example 2 (Fung's exponential): $\boldsymbol{\sigma = A(e^{B\varepsilon} - 1)}$
  - Q: What's the point of the - 1?
    - So that $\sigma(\varepsilon = 0) = 0$
  - $E_{tangent} = d\sigma/d\varepsilon = ABe^{B\varepsilon} = \sigma B + AB \rightarrow E_{tangent} \propto \sigma$

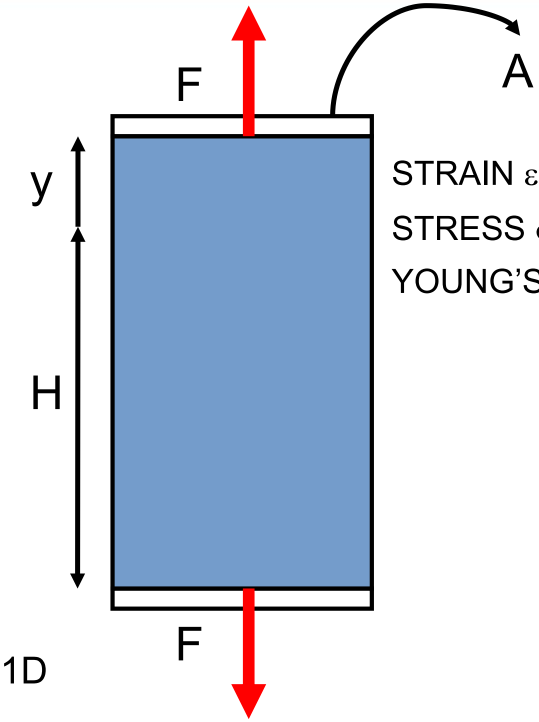
Characterization and modeling of nonlinearity





Kastelic et al. (1980); Frisen et al. (1969)

Complications of this simple picture



STRAIN $\varepsilon = y/H$

STRESS $\sigma = F_{ext}/A$

YOUNG'S MODULUS $E = \sigma/\varepsilon$

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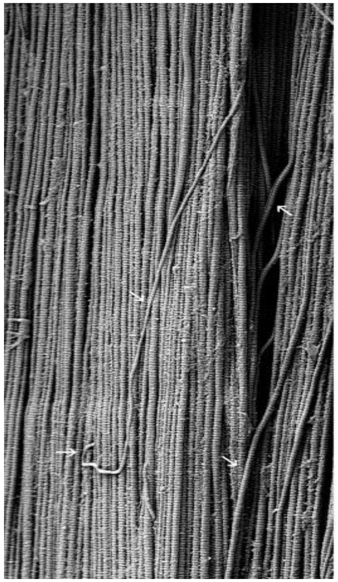
Would you expect the modulus for this…



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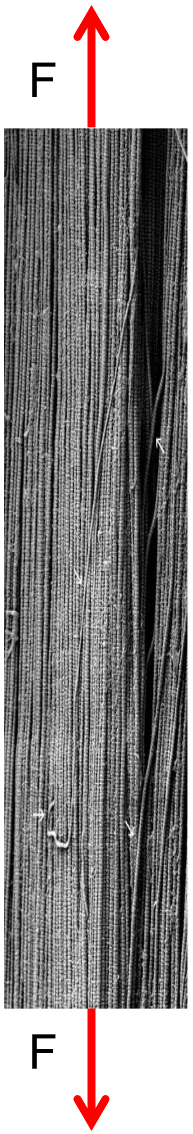
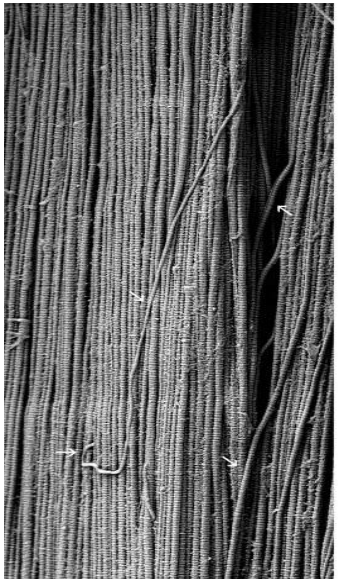


…to equal the modulus for this?



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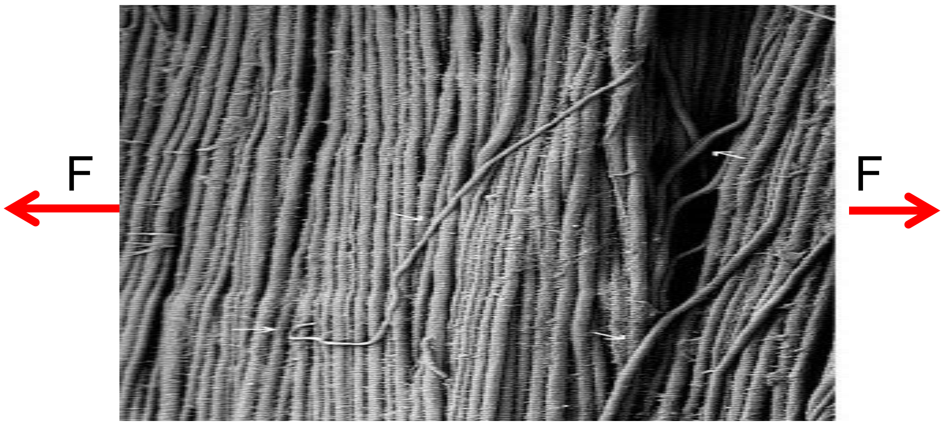


Biaxial testing

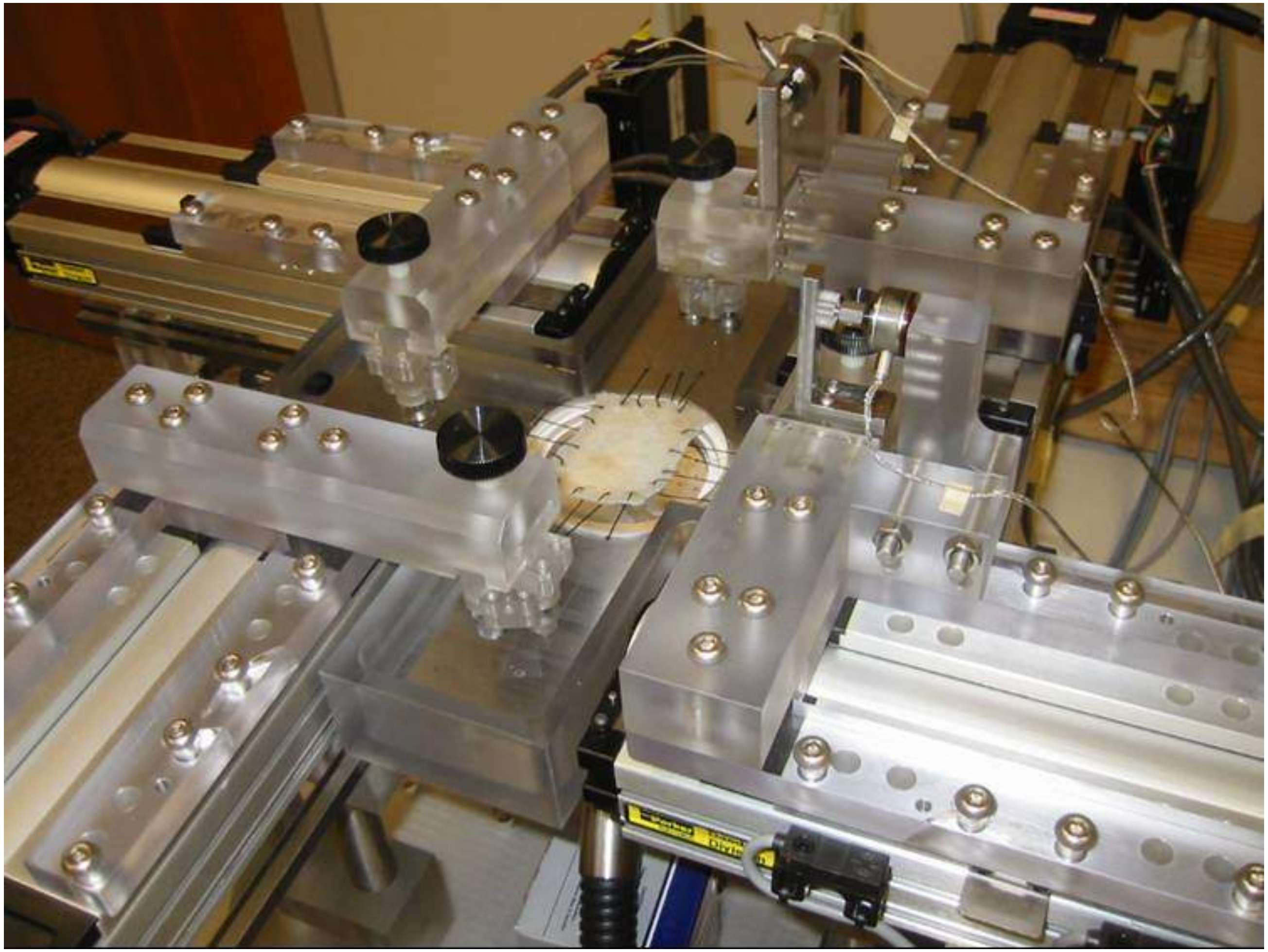
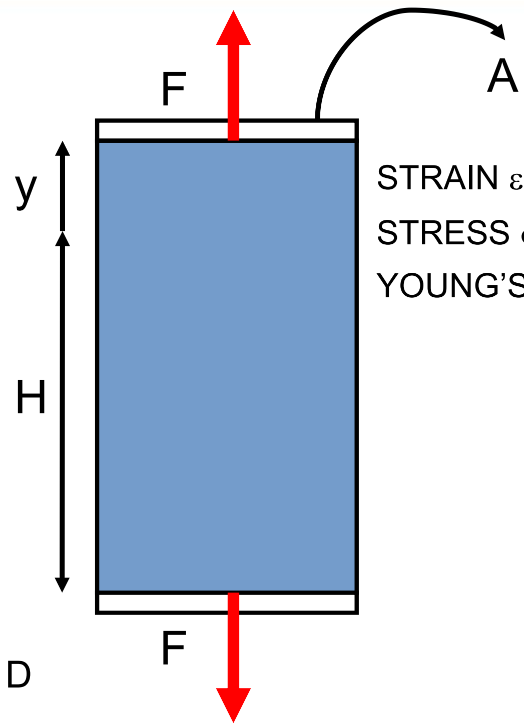


Photo from Michael Sacks's lab

Complications of this simple picture



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What is Viscoelasticity?

- **From R. Lakes:**

  "Viscoelastic materials are those for which the relationship between stress and strain depends on time."

- **From J. Maxwell:**

  "The state of the [viscoelastic] solid depends not only on the forces actually impressed on it, but on all the strains to which it has been subjected during its previous existences."

- **From R. M. Christensen:**

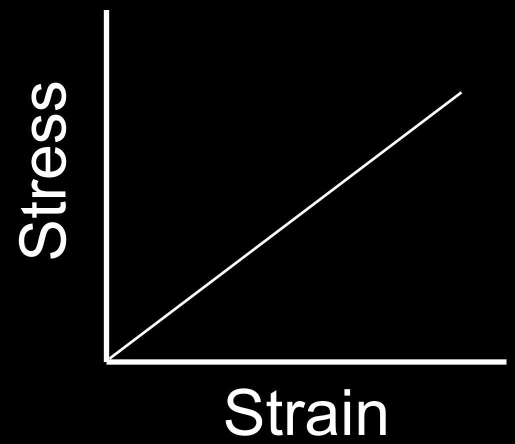
  "[Viscoelastic] materials possess a capacity to both store and dissipate energy."

- **From Wikipedia:**

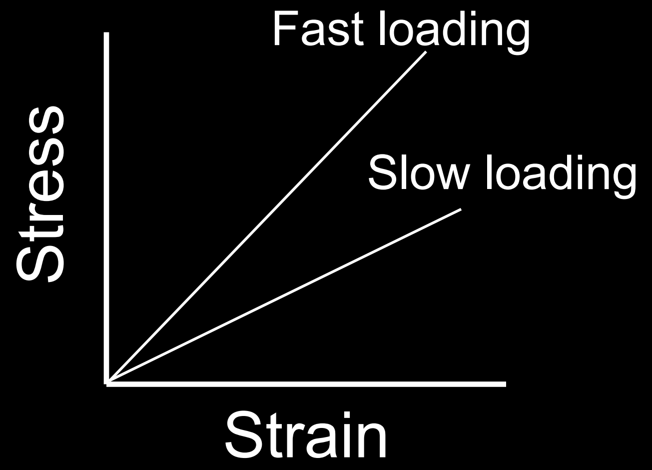
  "Viscoelasticity is the property of materials that exhibit both viscous and elastic characteristics when undergoing deformation."

Viscoelasticity: Rate dependence

ELASTIC MATERIAL

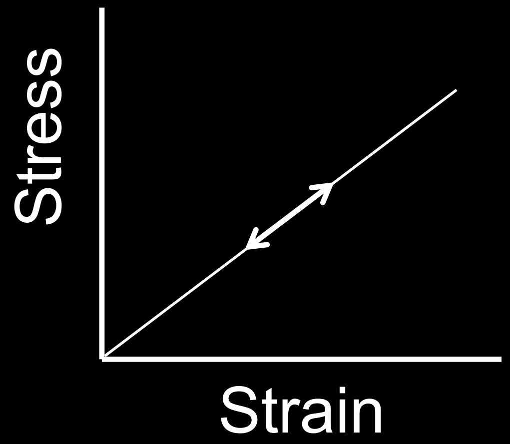


VISCOELASTIC MATERIAL

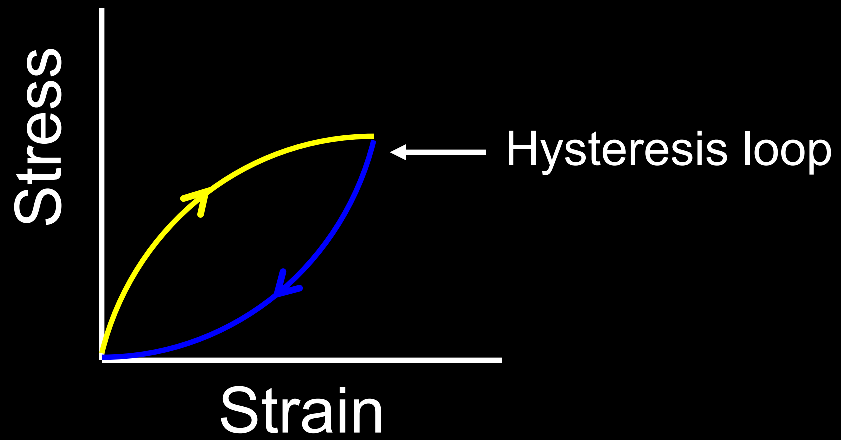


Viscoelasticity: Hysteresis

ELASTIC MATERIAL

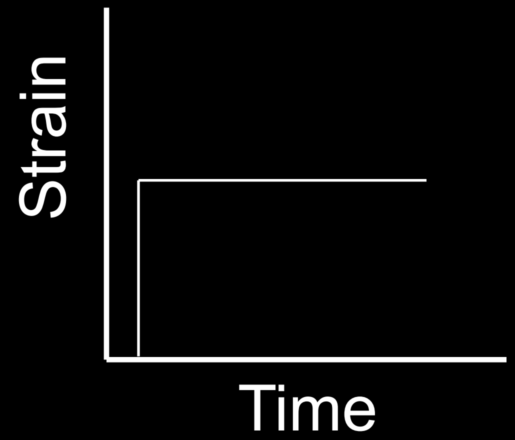


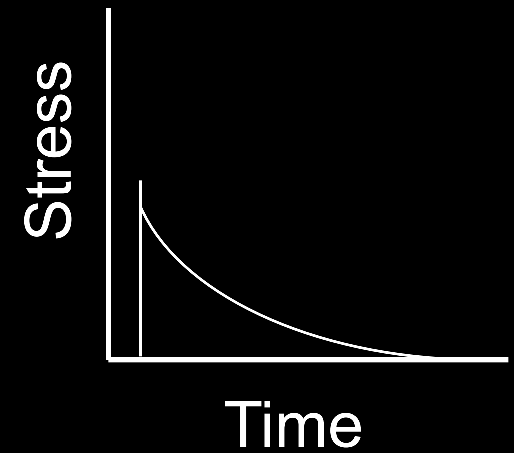
VISCOELASTIC MATERIAL



* Energy dissipated per cycle = area between yellow and blue curves

Viscoelasticity: Stress relaxation





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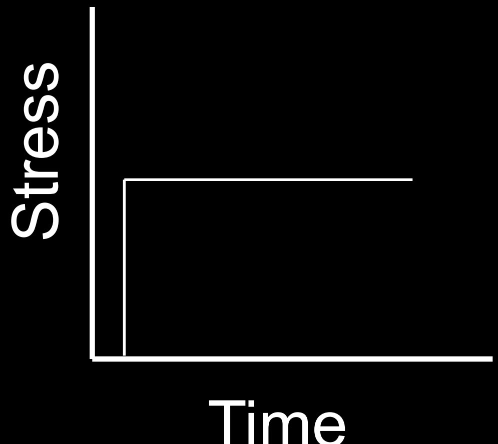
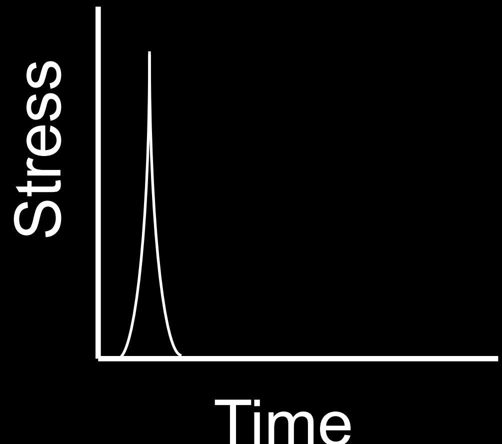
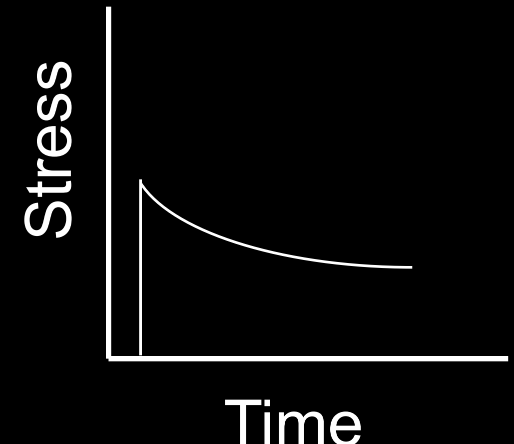


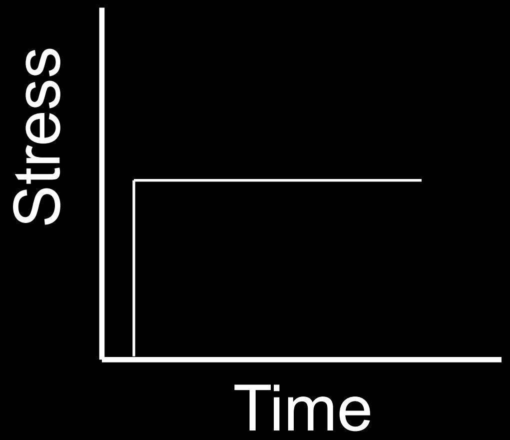
VISCOUS MATERIAL



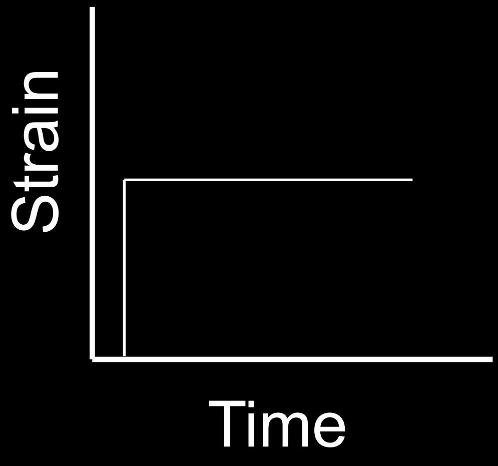
VISCOELASTIC MATERIAL



Viscoelasticity: Creep



ELASTIC MATERIAL



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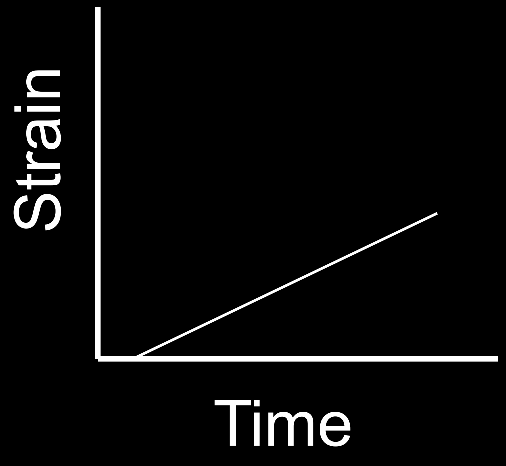

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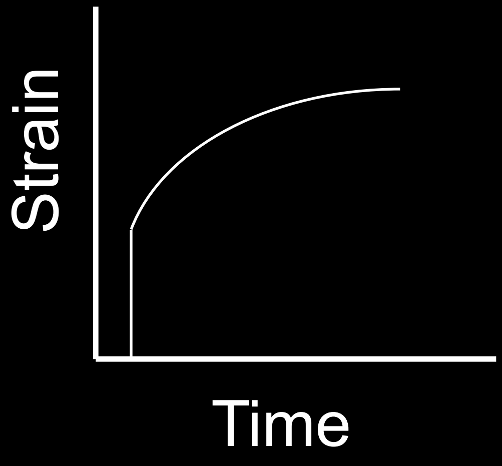
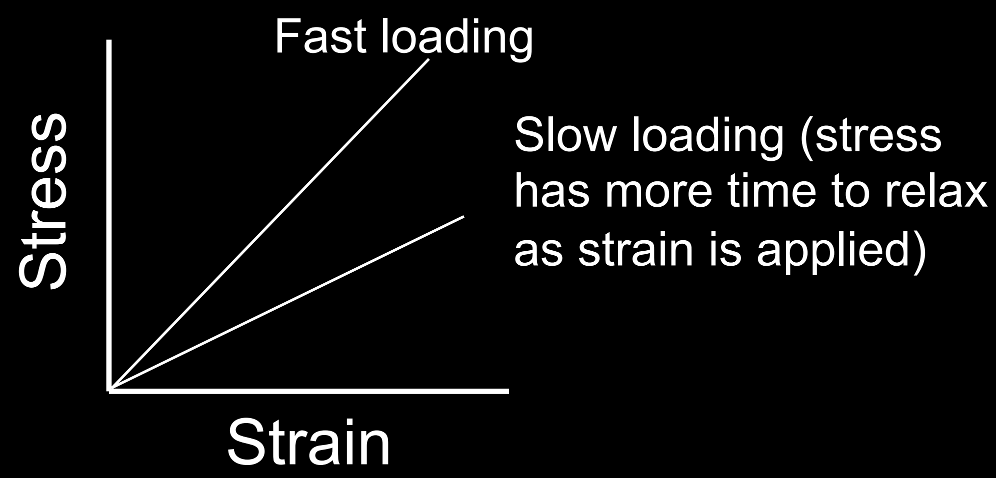
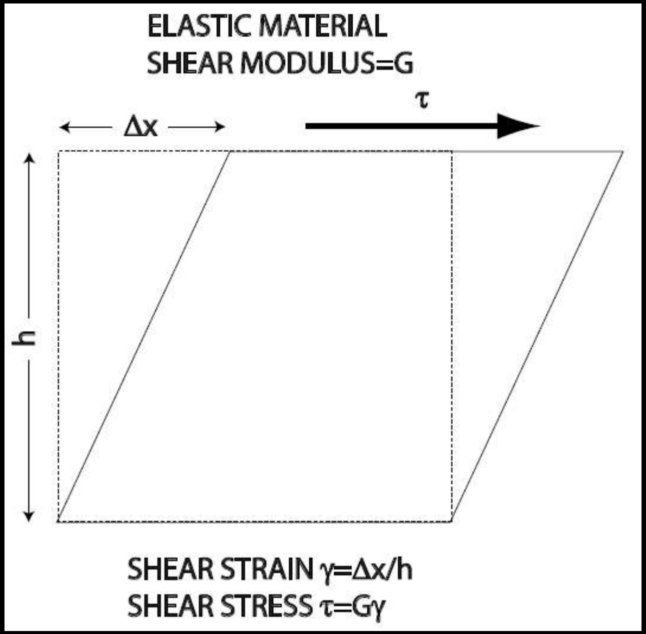


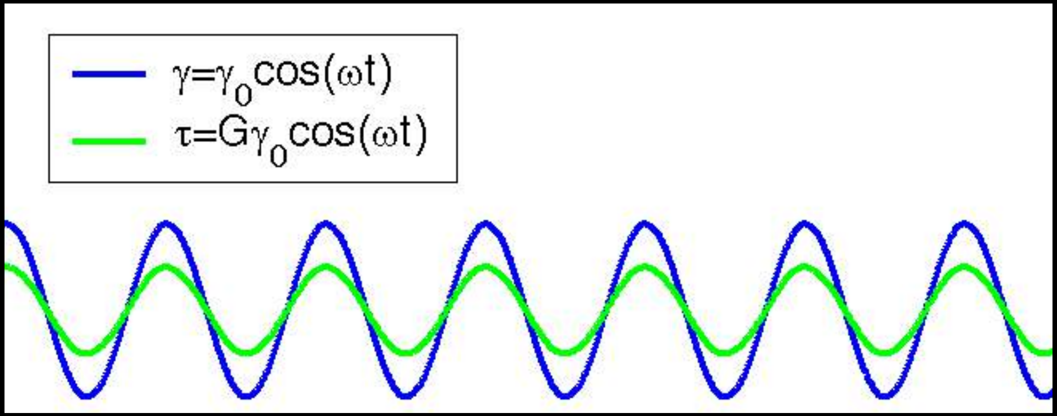
Relationships between viscoelastic phenomena

- All viscoelastic phenomena are related and are manifestations of the same general behavior
- For example, the rate dependence of the stress strain curve in a viscoelastic material is due to stress relaxing over time as strain is applied at a finite rate

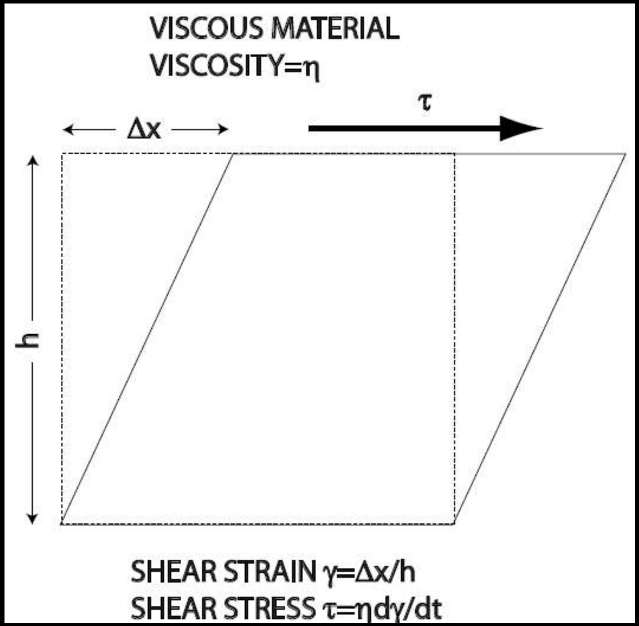


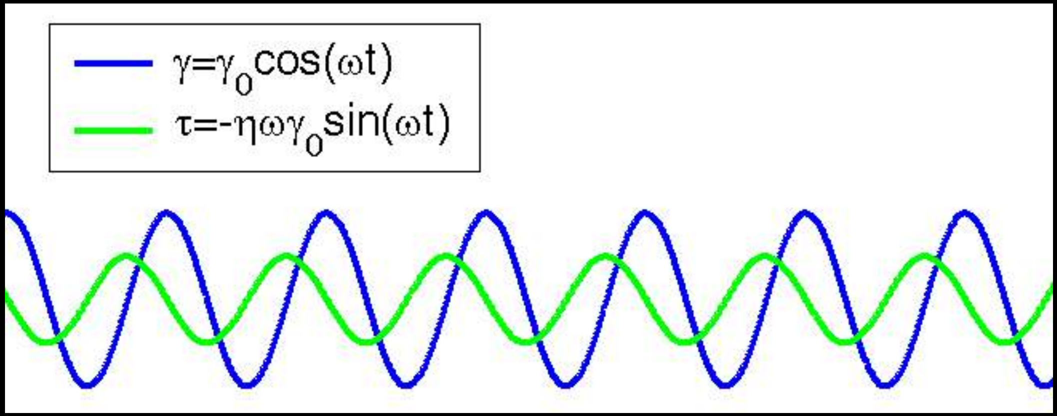
Cyclic loading of an elastic material



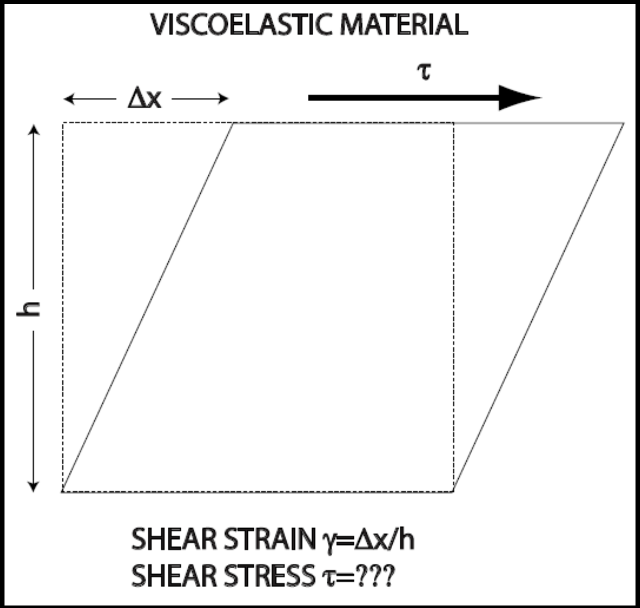


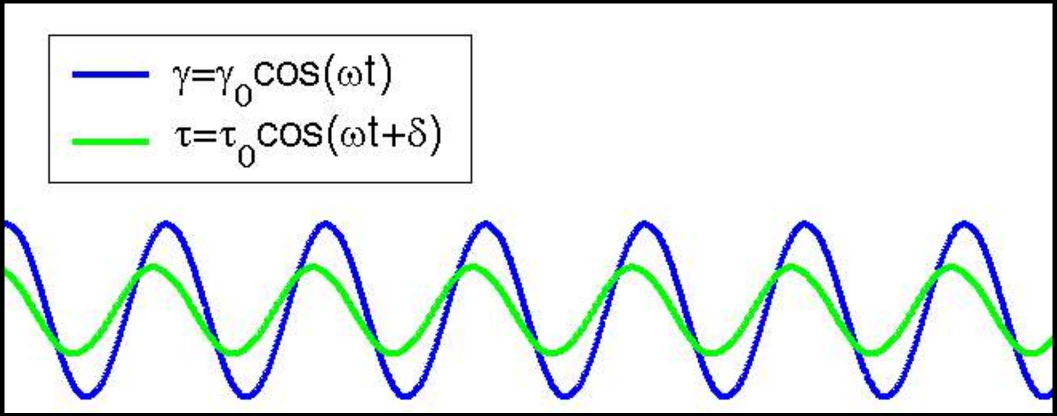
Cyclic loading of a viscous material





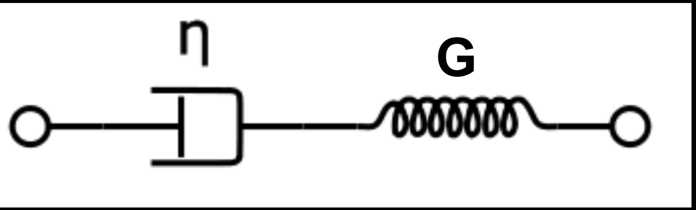
Cyclic loading of a viscoelastic material



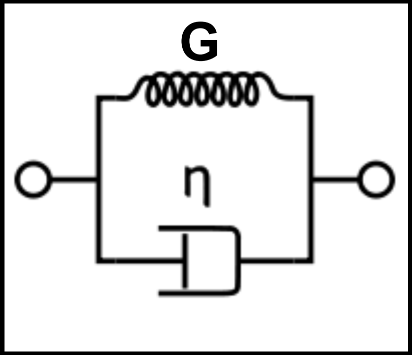


Viscoelastic materials: Modeling

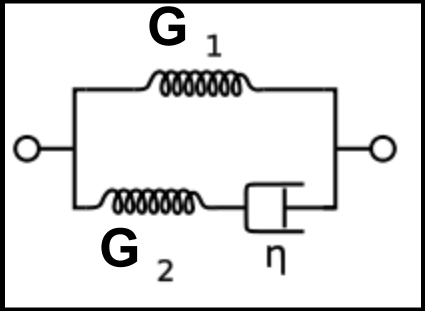
MAXWELL MODEL



VOIGT MODEL

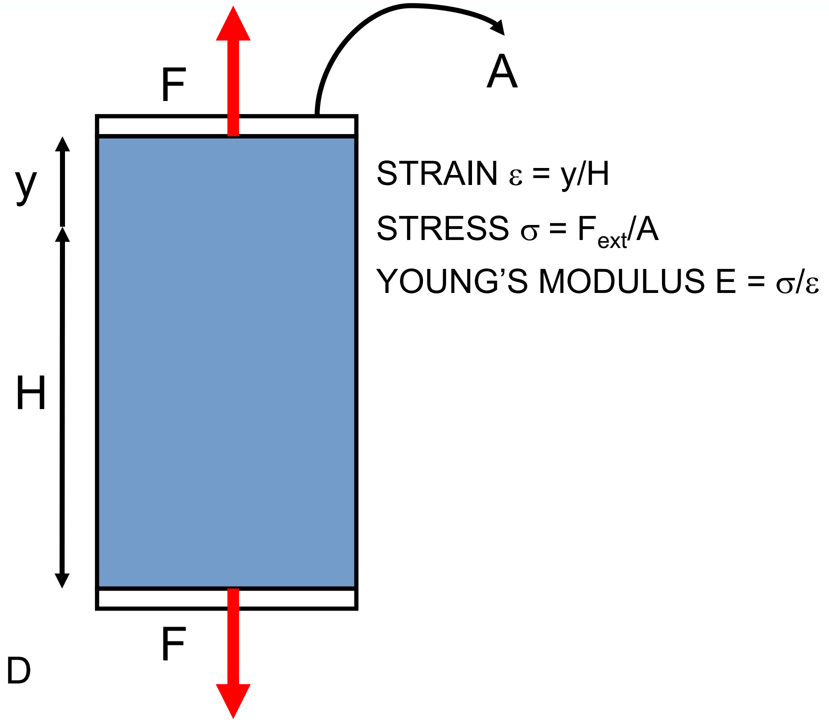


SLS MODEL



* Many more modeling strategies exist. Take BME 212 to learn more!

Complications of this simple picture



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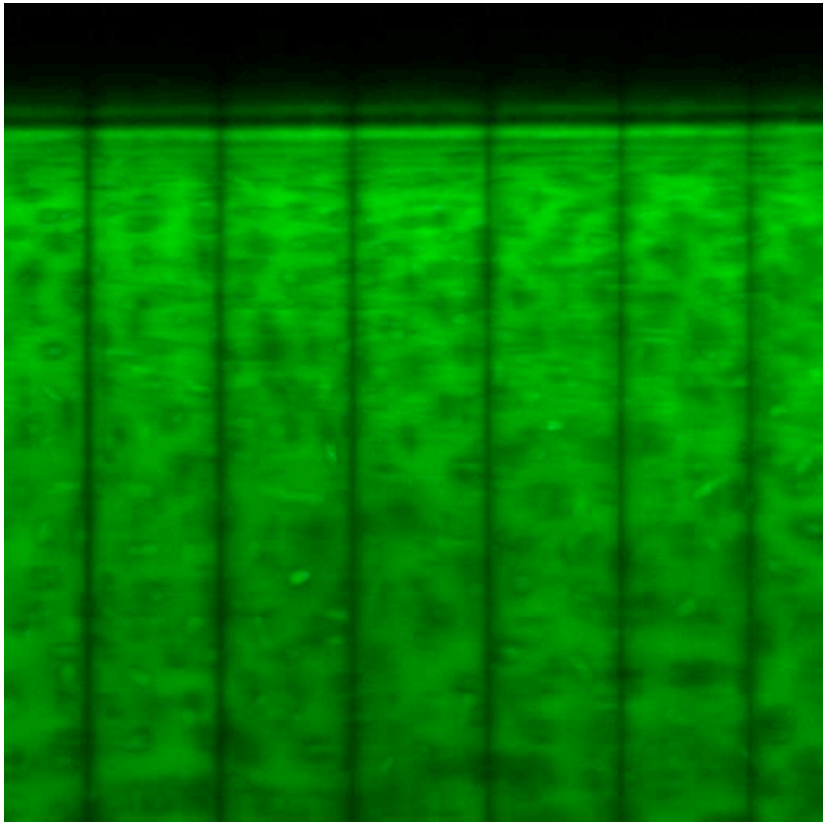
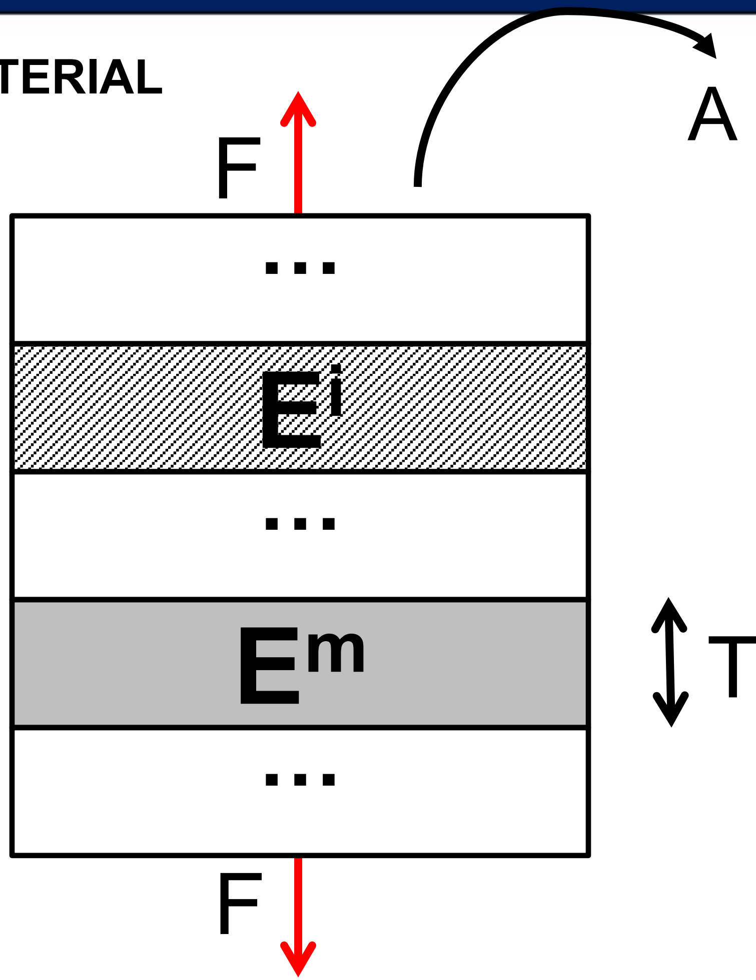
Shear deformation of articular cartilage



- Different regions of the tissue exhibit distinct strain patterns due to different material properties
- Bulk mechanical testing (apply shear strain to entire specimen, measure shear stress, infer G) cannot identify local variations in properties

Mechanics of heterogeneous materials

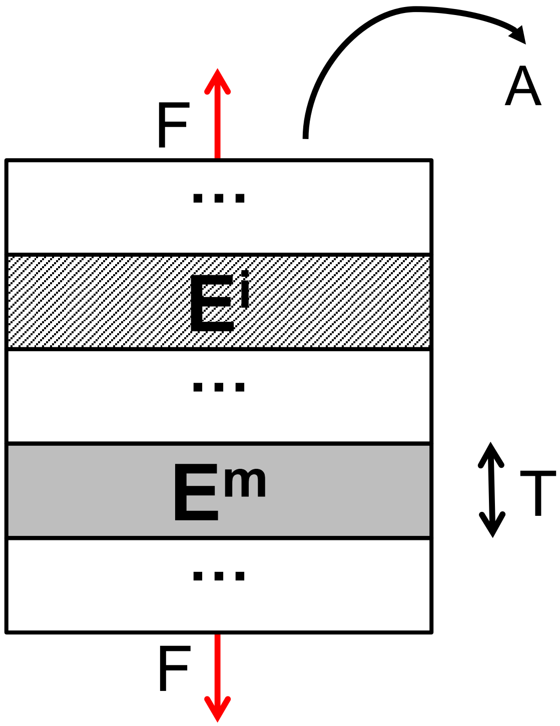
REUSS COMPOSITE MATERIAL



- *c → composite, m → matrix, i → inclusion*
- $T_i = n_i T$ (can be n_i different regions with E_i scattered throughout the specimen)
- $T_m = n_m T$
- $T_c = T_i + T_m$

How do we account for these complexities?

- *In a Reuss composite, F is the same in all layers. It is assumed that the stress σ=F/A is also the same in all layers (i.e., stress concentrations at the edges of the specimens and stress variations across the x direction are ignored)*
- $\sigma = E_i \varepsilon_i = \frac{E_i \Delta T_i}{T} \rightarrow \Delta T_i = \sigma T / E_i$
- But $\sigma = \varepsilon_c E_c = \frac{\sum_n \Delta T_n}{T_i + T_m} E_c$
- So $\frac{\sigma}{E_c} = \frac{n_i \sigma \frac{T}{E_i} + n_m \sigma \frac{T}{E_m}}{T_i + T_m}$
- But $\frac{n_i T}{T_i + T_m}$ is a volume fraction (ϕ_i)
- → $1/_{E_c} = \phi_i/_{E_i} + \phi_m/_{E_m}$



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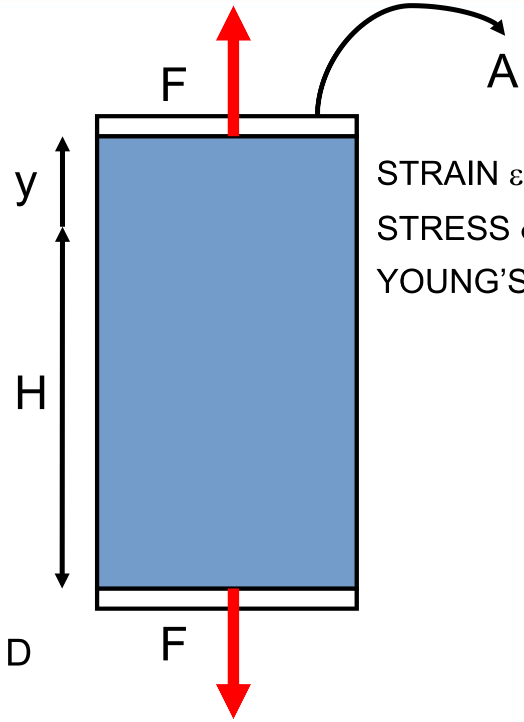

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Finite Element Analysis (FEA)

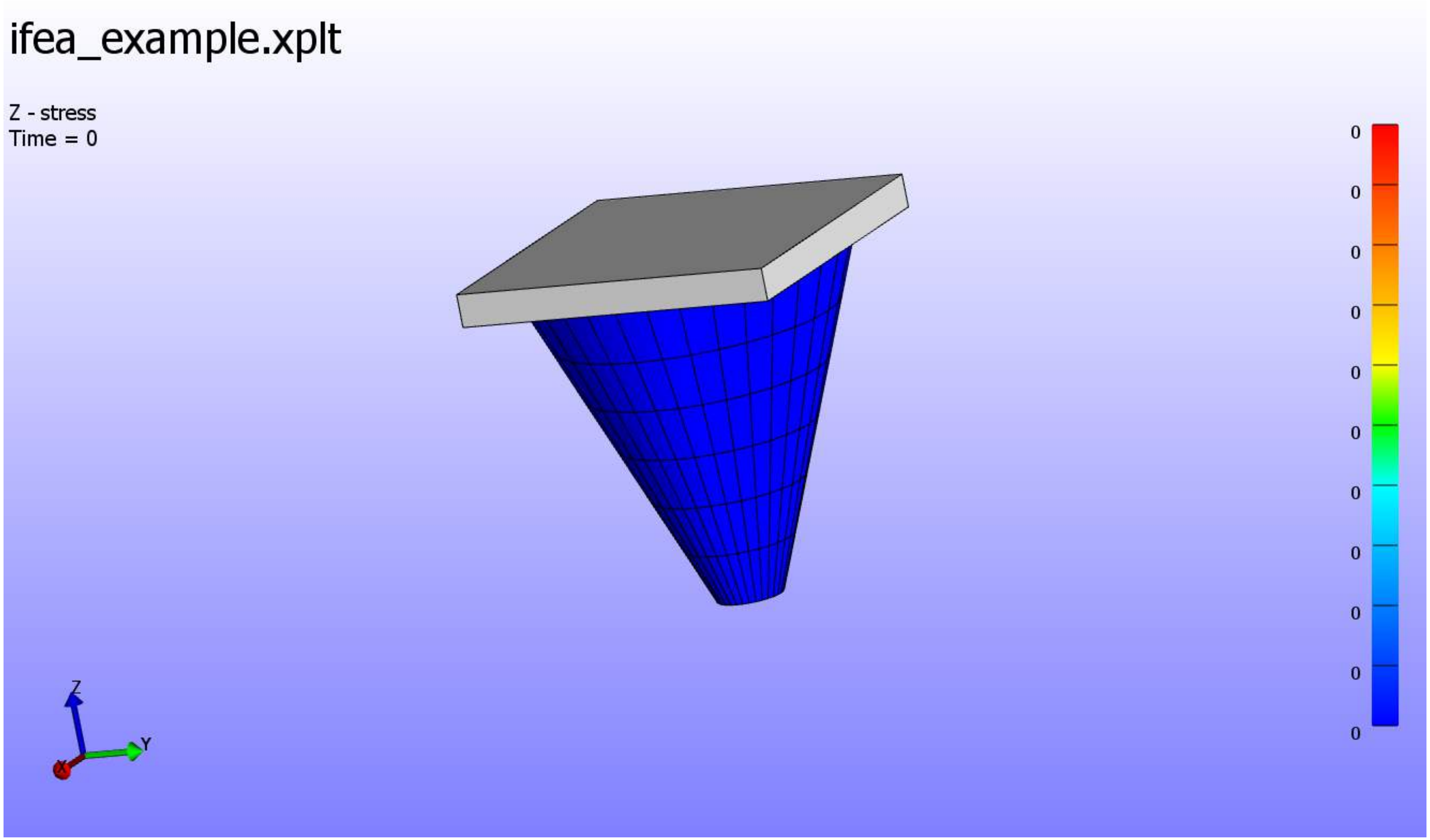
- The finite element method is a computationally efficient method of solving partial differential equations (PDE)
  - The PDE is written in a "variational" or "weak" form wherein the solution – a trial function u in a function space V – must satisfy an integral equation (across some spatial domain Ω) for any choice of a test function v in a function space $\hat{V}$
  - To approximate the solution in Ω, the spaces V and $\hat{V}$ are discretized such that u and v are taken to be finite dimensional (e.g., polynomials of degree 2 or less rather than polynomials of infinite degree or less)
  - Each spatial domain (Ω) defines an "element" and is prescribed by a 3D mesh. Intuitively, the smaller the size of each element, the closer the approximate solution is to the real solution
  - Boundary conditions are lumped into the definitions of V and $\hat{V}$
- The finite element method can be used to find solutions to heat transfer problems, mechanics problems, and a host of other problems involving PDEs

FEA ingredients

1. Model geometry (locations of nodes and how they connect)
2. Boundary conditions (boundary/body displacements/loads*)
3. Constitutive model (stress-strain relationship)
4. Material properties

* *May also need to consider boundary/body velocities and accelerations (kinematics) for some problems*

FEA example



Inverse finite element analysis

- Thus far, we have been focusing on forward finite element analysis
  - E.g., for a given geometry, boundary load (or boundary displacement), constitutive equation and material properties, what is the global and local deformation of the model?
- We can also use FEA to determine the material properties of a tissue, cell, etc. (e.g., for diagnosis of a disease) if we know the geometry and as much additional information as possible (e.g., boundary loads **and** boundary displacements).
  - This is a more complex procedure than forward FEA and involves iterative optimization. Essentially, the forward model is run with the known boundary displacements. If the measured boundary force is not attained, the material properties are changed and the model us run again. This procedure is continued until the simulation matches the experimentally measured boundary force.
  - We must assume a constitutive equation. If a complex constitutive model is chosen (i.e., with several material properties), it becomes more difficult to find a unique solution, as different combinations of material properties could yield similar behavior.

Inverse finite element analysis: Demo

- Scenario:
  - You have a conically-shaped biological tissue whose Young's modulus you want to know.
  - The bottom radius is 1 mm, the top radius is 5 mm and the height is 10 mm
  - You know it is an isotropic, elastic, incompressible material,.
  - In the lab, you subject the material to a uniaxial compression test (i.e., you compress it axially between two platens). For an applied deformation of -0.5 mm, the measured force on the material is -100 N.
  - You don't know the analytical relationship between force, deformation and Young's modulus in a material with this geometry, so you decide to use inverse FEA

Inverse finite element analysis: Demo

- First, run the forward model for a conically-shaped material with arbitrary Young's modulus (say, 100 MPa) subjected to the deformation you applied in the lab
- If the force (output from the forward mode) is greater in magnitude than -100 N, decrease the Young's modulus in the model and rerun the simulation.
- If the force (output from the forward mode) is lower in magnitude than -100 N, increase the Young's modulus in the model and rerun the simulation.
- Repeat similar steps until the rigid force is -100 N. Then the modulus you prescribed is the modulus of the material.